MAT291: Introduction to Mathematical Physics

Student Textbook & Lecture Notes

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PREFACE

These notes have been collected from those I took from the course textbook and instructor-provided readings throughout the semester. These notes are not a full representation of the course's content, and **there may be information that I noted down incorrectly**. I am posting these notes to my website so that future MAT291 students can take whatever they need from it. That being said, the content in these notes was taken from:

1. *Calculus: Early Transcendentals*, Third Edition, Briggs et al., and
2. Professor Francis Dawson's Class Notes and Reading Assignments for MAT291.

I also have a GitHub repository containing all these notes on a chapter-by-chapter basis sorted by week. It can be found at `github.com/arnav-patil-12/mat291-notes`.

CONTENTS

Part I

Vectors and Vector Spaces

CHAPTER 1

LINES AND PLANES IN SPACE

1 Lines in Space

We can define a unique line in $\mathbb{R}^3$ using either two distinct points or one point and a direction. We can use these properties to derive two different descriptions of lines: parametric equations, and vector equations.

Component Form

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

Vector Form

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

This gives us $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$.

2 Distance from a Point to a Line

Consider a line l given by $\vec{r} = \vec{r}_0 + t\vec{v}$. Our goal is to find the distance d between Q and l. We form another perpendicular line from Q to Q' on l to make a right-angle triangle PQQ', thus, the shortest distance from Q to l is the distance from Q to Q'.

We know that $d = |\vec{PQ}|\sin\theta$, where θ is the angle between $\vec{v}$ and $\vec{PQ}$. We have:

$$|\vec{v} \times \vec{PQ}| = |\vec{v}||\vec{PQ}|\sin\theta = |\vec{v}|d$$

From which we can derive:

$$d = \frac{|\vec{v} \times \vec{PQ}|}{|\vec{v}|}$$

3 Equations of Planes

A plane is a flat surface that infinitely extends in all directions. Three points, where not all points are on the same line, determine a unique plane in $\mathbb{R}^3$. A plane can also be determined by one point on the plane and a nonzero vector orthogonal to it. This vector is called a **normal vector** and specifies the orientation of the plane.

We can formally define a plane as: Given a fixed point P_0 and a nonzero vector $\vec{n}$, the set of points P for which $\vec{P_0P}$ is orthogonal to $\vec{n}$ is called a plane. Just as a slope defines the orientation of a line in $\mathbb{R}^2$, a normal vector determines the orientation of a plane in $\mathbb{R}^3$.

The equation of a plane in $\mathbb{R}^3$ can be given by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \text{ OR } ax + by + cz = d$$

Note: a vector $\vec{n} = \langle a, b, c\rangle$ is used to describe a plane by specifying a direction orthogonal to the plane. On the other hand, a vector $\vec{v} = \langle a, b, c\rangle$ is used to describe a direction parallel to the line.

4 Parallel and Orthogonal Planes

Normal vectors tell us about the orientation of planes. In particular, there are two cases of interest: one where the planes are parallel, and one where the planes are orthogonal relative to each other. If $\vec{n}_1$ and $\vec{n}_2$ are parallel, then the planes are parallel. If $\vec{n}_1 \cdot \vec{n}_2 = 0$, then the planes are orthogonal.

We will take a look at an example to solidify our understanding. Find an equation of the line of intersection of the planes $Q : x+2y+z = 5$ and $R : 2x+y-z = 7$. First note $\vec{n}_1 = \langle 1, 2, 1\rangle$ and $\vec{n}_2\langle 2, 1, -1\rangle$ are not multiples of each other. Thus, the planes are not parallel and they must intersect in a line l. To find l, we need a point on l and a vector pointing in the direction of l. Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect. By setting $z = 0$, we find a point that lies on both planes and on the xy plane ($z = 0$).

$$x + 2y = 5$$
$$2x + y = 7$$

After solving this system, we see that $x = 3$ and $y = 1$. We see that $(3, 1, 0)$ is a point on l. Next, we need to find a vector parallel to l. Because l lies in Q and R, it is orthogonal to $\vec{n}_Q$ and $\vec{n}_R$. The cross product is a vector parallel to l, which in this case is $\langle -3, 3, -3\rangle$.

Therefore, any point on the line $\vec{r} = \langle 3, 1, 0\rangle + t\langle -3, 3, -3\rangle$ satisfies the equations of both planes. In other words, any point (x, y, z) satisfying $x = 3 - 3t$, $y = 1 + 3t$, $z = -3t$ also satisfies both plane equations.

CHAPTER 2

CYLINDERS AND QUADRIC SURFACES

In **Section 13.5**, we discussed how lines and planes are described by parametric linear equations. In this section, we will look at the geometry of three-dimensional objects described by quadratic equations in three variables. These are *quadric surfaces*.

1 Cylinders and Traces

In the vernacular, we use cylinder to describe the curved wall of a paint can, for example. In the context of this textbook, we will use **cylinder** to refer to a a surface that is parallel to a line, specifically in this text, we focus on cylinders parallel to one of the coordinate axes.

An example in $\mathbb{R}^3$ is $y = x^2$, which you will notice does not contain z. This means z is arbitrary and can take on any value.

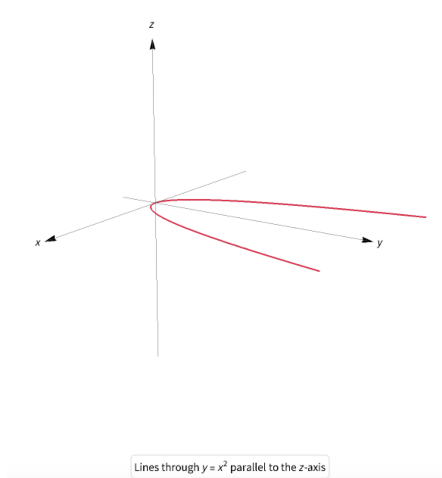


Figure 2.1: Cylinder parallel to the z-axis

A surface's **trace** is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes.

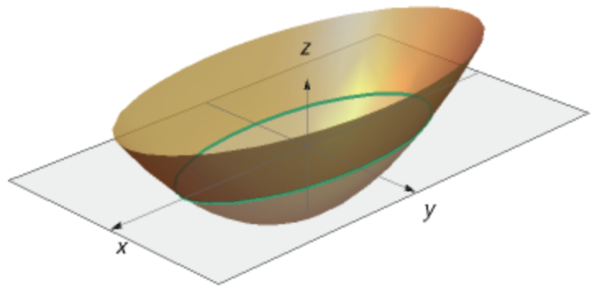


Figure 2.2: xy-trace given by the green line

2 Quadric Surfaces

Quadric surfaces are given by the general quadratic equation in three variables:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where not all of A, B, C, D, E, F are zero.

In this text, we will not conduct a detailed study of quadric surfaces, but there are some of particular interest, which have various practical uses. For examples, paraboloids have the same reflective properties as their 2D equivalents, and are used to design satellite dishes and mirrors in telescopes. Hyperboloids inspire the shape of nuclear plant cooling towers. Lastly, ellipsoids help design water tanks and gears.

Here are some tips to keep in mind when drawing quadric surfaces:

1. **Intercepts** – Determine the points where the surface intersects the coordinate axes. Do this by setting x, y, and z to zero two-at-a-time and solving for the third.

2. **Traces** – Finding traces helps visualizes the surface. By setting $z = 0$, we can find all points parallel to the xy-plane.

3. **Completing the figure** – Sketch at least two more traces, say $z = 1$ and $z = -1$, then draw curves that pass through all traces to complete the surface.

Part II

Functions of Several Variables

CHAPTER 1

GRAPHS AND LEVEL CURVES

1 Functions of Two Variables

Concepts related to functions of several variables is explained intuitively using the case of two independent variables, and this understanding can then be generalized for functions of more than two variables. In general, functions of two variables can be written **explicitly** as:

$$z = f(x, y)$$

or **implicitly** as:

$$F(x, y, z) = 0$$

The idea of domain and range in these functions is the same as in functions of one variable.

A function $z = f(x, y)$ assigns a point (x, y) in $\mathbb{R}^2$ to unique number z in $\mathbb{R}$.

2 Graphs of Functions of Two Variables

A graph of a function $f(x, y)$ is the set of points (x, y, z) that satisfies $z = f(x, y)$. We may also write this set of points as $(x, y, f(x, y))$ or $F(x, y, z) = 0$.

Functions of two variables must pass the vertical line test, i.e., a relation $F(x, y, z) = 0$ is a function if and only if every line parallel to the z-axis intersects the graph at only one point.

2.1 Level Curves

Functions of two variables are represented in $\mathbb{R}^3$ as surfaces, which can be represented in a manner similar to a contour map.
Consider a surface defined by $z = f(x, y)$, and imagine walking on this surface such that your elevation is always a constant value $z = z_0$. This path is called a **contour curve**. This curve in the xy-plane is called a **level curve**. A contour curve is indeed a **trace** in the plane $z = z_0$. A level curve is not necessarily a trace, it may consist of a single point, a group of points, or a group of curves.

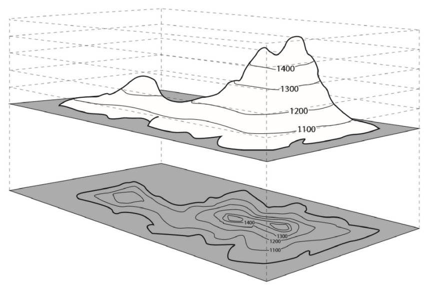


Figure 1.1: How to read a contour map

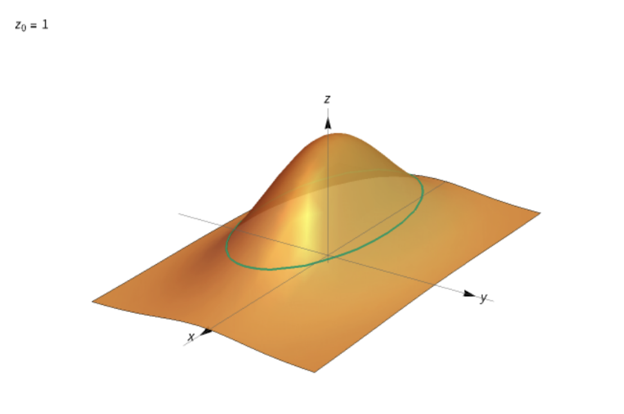


Figure 1.2: Exmaple of a level curve

3 Applications of Functions of Two Variables

3.1 A Probability Function of Two Variables

On a particular day, the fraction of students with the flu is r such that $0 \leq r \leq 1$. If you have n encounters with students, the probability of meeting at least one infected student is $p(n, r) = 1 - (1 - r)^n$.

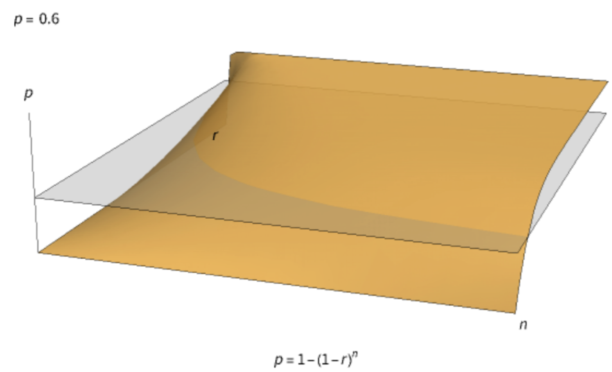


Figure 1.3: Probability expressed as a graph

r is restricted to $[0, 1]$ as it's a fraction of the population, and n is any non-negative integer. Since $0 \leq r \leq 1$, we have $0 \leq 1 - r \leq 1$. And since n cannot be negative, we have $0 \leq (1 - r)^n \leq 1$, which makes the range of the function $[0, 1]$, consistent with the fact that it is a probability.

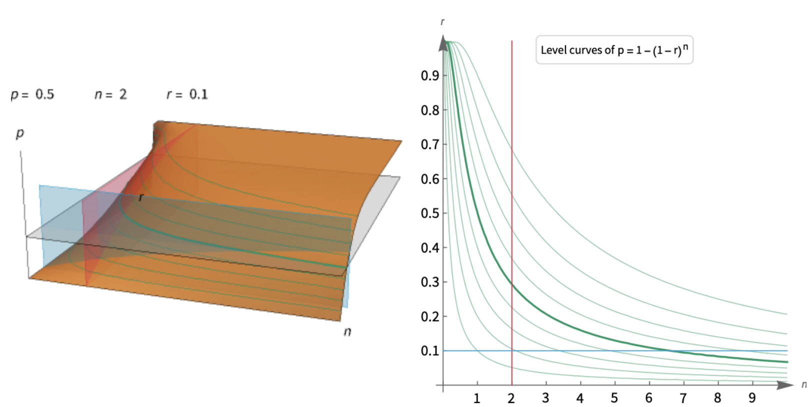


Figure 1.4: Level curve graph of the problem

4 Functions of More Than Two Variables

The properties of functions of two variables extend to functions of more than two variables. For example, a function in three variables is defined explicitly as $w = f(x, y, z)$ and implicitly as $F(x, y, z, w) = 0$.

We can more formally define a function with n independent variables as follows: the function $x_{n+1} = f(x_1, x_2, ..., x_n)$ assigns a unique real number x_{n+1} to each point $(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$.

5 Graphs of Functions of More Than Two Variables

Graphing functions of two independent variables required a three-dimensional coordinate system, which is the limit of ordinary graphing functions. There are two approaches:

The idea of level curves can be extended. Given function $w = f(x, y, z)$, we can set $w = 0$ as a constant. Then, the surface formed by all points for which $f(x, y, z) = 0$ holds becomes a **level surface**.

Another approach is to use colour to portray the fourth dimension. See the image below:

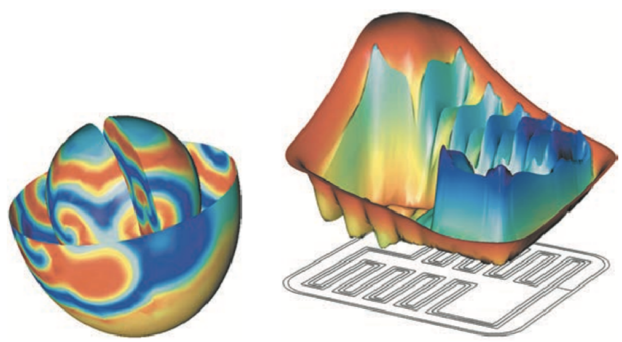


Figure 1.5: Using a colour gradient as a representation of the fourth dimension

CHAPTER 2

LIMITS AND CONTINUITY

1 Limit of a Function of Two Variables

A function f of two variables has a limit L as $P(x,y)$ approaches a fixed point $P_0(a,b)$ if $|f(x,y)-L|$ can become arbitrarily small for all P. If this limit exists, we can write:

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{P\to P_0} f(x,y) = L$$

We can construct a more formal definition of a limit of a function of two variables: The function f has the limit L as $P(x,y)$ approaches $P_0(a,b)$, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{P\to P_0} f(x,y) = L$$

if, given any $\varepsilon > 0$, there exists a $\sigma > 0$ such that $|f(x,y)-L| < \varepsilon$ whenever:

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \sigma$$

Therefore, the limit only exists if $f(x,y)$ approaches L as P approaches P_0 along all possible paths.

All the limit laws apply as they do for functions in a single variable.

2 Limits at Boundary Points

Let R be a region in $\mathbb{R}^2$. An **interior point** of R lies entirely in R, meaning we can construct a disk centred at that point containing only points within R also. A **boundary point** of R lies on the edge of R such that every disk centred at that point contains at least one point outside and inside R. We may translate these definitions to $\mathbb{R}^3$ by replaces *disk* with *ball*.

With these definitions, we may also define **open sets** as regions consisting entirely of interior points, and **closed sets** as regions containing all of their boundary points.

Let us consider a few examples:

Example: limits at boundary points

Evaluate $\lim\limits_{(x,y)\to(4,1)} \dfrac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$.

Solution: Points in the domain must satisfy $x \geq 0$ and $y \geq 0$ and $x \neq 4y$. Due to the last condition, we see that the point $(4,1)$ lies on the boundary of the domain.

$$\begin{aligned}
\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}} &= \lim_{(x,y)\to(4,1)} \frac{(xy-4y^2)(\sqrt{x}+2\sqrt{y})}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}+2\sqrt{y})} \\
&= \lim_{(x,y)\to(4,1)} \frac{y(x-4y)(\sqrt{x}+2\sqrt{y})}{x-4y} \\
&= \lim_{(x,y)\to(4,1)} y(\sqrt{x}+2\sqrt{y}) \\
&= 4.
\end{aligned}$$

Example: nonexistence of a limit

Investigate the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2+y^2}$.

Solution: The domain of this function is $D : \{(x,y)|(x,y) \neq (0,0)\}$, meaning the limit is at a boundary point *outside* of the domain. Suppose we let (x,y) approach $(0,0)$ along the line $y = mx$; we can substitute $y = mx$. Thus:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} &= \lim_{x\to 0} \frac{(x+mx)^2}{(x^2+m^2x^2)} \\
&= \lim_{x\to 0} \frac{x^2(1+m)^2}{x^2(1+m^2)} \\
&= \frac{(1+m)^2}{1+m^2}.
\end{aligned}$$

where the constant m determines the direction of approach to $(0,0)$. In other words, the function approaches different values as $(x,y) \to (0,0)$, depending on the value of m.

The Two-Path Test for Nonexistence of Limits – If (x,y) approaches two different values as $(x,y) \to (a,b)$, then the limit does not exist.

3 Continuity of Functions of Two Variables

The definition of continuity for functions of two variables is the same as the definition for functions of one variable.

The function f is continuous at the point (a,b) given:

1. f is defined at (a, b),
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, and
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

3.1 Composite Functions

Recall that a composition of a continuous function (in a single variable) is also continuous. If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is also continuous at (a, b).

4 Functions of Three Variables

Work done with limits and continuity of functions of two variables extends of three+ variables. Limits of rational and polynomial functions may be evaluated by directly substituting at all points within their domains. Compositions of continuous functions $f(g(x, y, z))$ are continuous at points at which $g(x, y, z)$ is within the domain of f.

CHAPTER 3

PARTIAL DERIVATIVES

1 Limit of a Function of Two Variables

Suppose we are standing on the surface below, at the point $P(0, 0, f(0, 0))$. If we walk east-west, we will walk uphill, and if we walk north-south, we will walk downhill. Essentially, the function value changes at a different rate in every direction we walk from P. So, how do we define the slope (or rate of change) at a given point on this surface? The answer is partial derivatives.

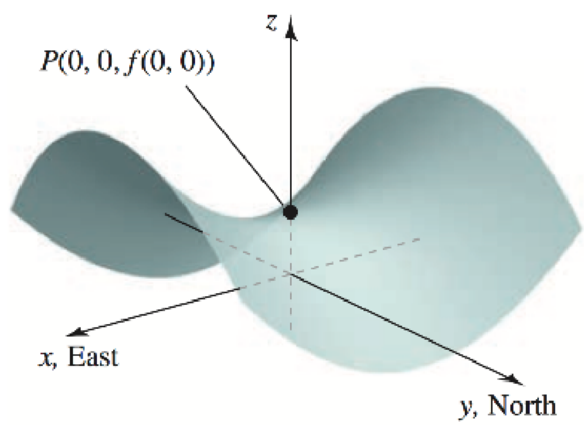


Figure 3.1: Saddle Graph

Partial derivatives are when we hold all but one independent variable fixed, then compute an ordinary derivative with respect to it.

Let's say we are moving on the surface $z = f(x, y)$ starting at the point $(a, b, f(a, b))$ in a way such that $y = b$ is fixed and only x is allowed to vary. The resulting path is a trace on the surface with vertical plane $y = b$. This path is described by $z = f(x, b)$ which is a function of the single variable x. We can calculate the slope of this curve as the ordinary derivative of $f(x, b)$ with respect to x. This derivative is called the partial derivative of f with respect to x, and is denoted as $\frac{\partial f}{\partial x}$ or f_x. We can define the limit:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

Similarly, we can move along $z = f(x,y)$ in such a way that $x = a$ is fixed and y varies. Now, the result is a trace described by $z = f(a,y)$, the intersection of the surface and the plane $x = a$. Now, we can take the ordinary derivative of this with respect to y.

Definition: The partial derivative of f with respect to x at the point (a,b) is:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

The partial derivative of f with respect to y at the point (a,b) is:

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

provided these limits exist.

Partial derivatives may be denoted in any of the following ways:

$$\frac{\partial f}{\partial x}(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b)$$

Partial derivatives from the definition

Example: Suppose $f(x,y) = x^2y$. Use the limit definition of partial derivatives to compute $f_x(x,y)$ and $f_y(x,y)$.

Solution: We compute the partial derivatives at an arbitrary point (x,y) in the domain. The partial derivative with respect to x is:

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2y - x^2y}{h} \\ &= \lim_{h\to 0} \frac{(x^2+2xh+h^2-x^2)y}{h} \\ &= \lim_{h\to 0} (2x+h)y \\ &= 2xy \end{aligned}$$

The partial derivative with respect to y is:

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{x^2(y+h)-x^2y}{h} \\ &= \lim_{h\to 0} \frac{x^2(y+h-y)}{h} \\ &= x^2 \end{aligned}$$

There is a shortcut revealed to evaluate partial derivatives: to compute the partial derivative of f with respect to x, we treat y as a constant and differentiate:

$$\frac{\partial}{\partial x}(x^2y) = y\frac{\partial}{\partial x}(x^2) = 2xy$$

We can apply the same shortcut to differentiate with respect to y:

$$\frac{\partial}{\partial y}(x^2y) = x^2\frac{\partial}{\partial y}(y) = 2x$$

2 Higher-Order Partial Derivatives

Just as we can take second, third (and so on) derivatives of functions of one variable, we can have higher-order partial derivatives as well. Say we take the partial derivative f_x of a function f, then we can further differentiate f_x with respect to x or y. This means there are four possible second-order derivatives to f.

2.1 Equality of Mixed Partial Derivatives

A mixed partial derivative is taken with respect to x the first time and y the next, or vice versa. Essentially, do not differentiate with respect to the same variable both times.

Theorem: Clairaut Equality of Mixed Partial Derivatives – Assume f is defined on an open set D of $\mathbb{R}^2$, and f_{xy} and F_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

3 Functions of Three Variables

Everything covered thus far about partial of derivatives of functions with two variables carries over to functions of three or more variables.

4 Differentiability

Although we have learned to differentiate partial derivatives of a function of several variables, we have not yet covered what it means for a function to be *differentiable* at a point. Recall that a function f of one variable is differentiable at $x = a$ provided:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

If f can be differentiated at a, that means there is a smooth curve at a (no jumps, corners, or cusps). The function also has a unique tangent line at that point with value $f'(a)$.

For a function of several variables, the surface should be smooth at that point, and there should also exist something analogous to a tangent line. We define the quantity:

$$\varepsilon = \frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)$$

where ε is a function of Δx. We can multiply both sides by Δx to give:

$$\varepsilon \Delta x = f(a + \Delta x) - f(a)\Delta x$$

which when further rearranged gives the change in function $y = f(x)$:

$$\Delta y = f(a + \Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x$$

Definition: Differentiability – The function $z = f(x, y)$ is differentiable at (a, b) provided the change $\Delta z = f(a+\Delta x, b+\Delta y) - f(a, b)$ equals $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$. A function is differentiable on an open region R if it is differentiable at every point on R.

Theorem: Conditions for Differentiability – Suppose a function f has partial derivatives f_x and f_y defined on an open set containing (a, b) with f_x and f_y being continuous at (a, b). Then, f is differentiable at (a, b).

CHAPTER 4

THE CHAIN RULE

1 The Chain Rule with One Independent Variable

If we have y is a function of u and u is a function of t, then we have:

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt}$$

Now we consider composite functions of the form $z = f(x, y)$ where x and y are functions of t. Then, what is dz/dt. We can illustrate the relationships among these variables using a tree diagram.

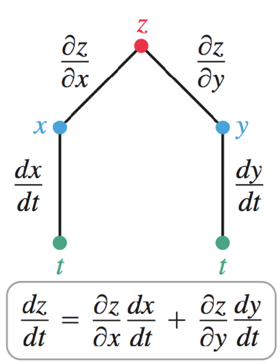


Figure 4.1: Tree Diagram

Theorem 15.7 – Chain Rule (One-Independent Variable)

Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Some comments are in order:

- With $z = f(x(t), y(t))$, the dependent variable is z and the sole independent variable is t.
- x and y are intermediate variables.
- The above theorem generalizes directly to functions of more than two variables.

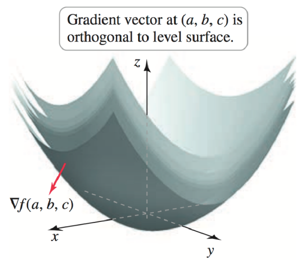


Figure 4.2: Diagram Tree for a Function of Three Variables

2 The Chain Rule with Several Independent Variables

Suppose z depends o two independent variables s and t. Once again, we can use a diagram tree to organize the relationships among variables.

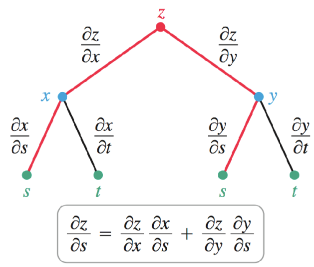


Figure 4.3: Chain Rule with Multiple Variables

Theorem 15.8 – Chain Rule (Two Independent Variable)

Let z be a differentiable function of x and y, who are in turn functions of s and t. Then:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

3 Implicit Differentiation

Theorem 15.9 – Implicit Differentiation

Let F be differentiable on its domain and suppose $F(x, y) = 0$ defined y as a differentiable function of x. Provided $F_y \neq 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

CHAPTER 5

DIRECTIONAL DERIVATIVES AND THE GRADIENT

1 Directional Derivatives

Let $(a, b, f(a, b))$ be a point on the surface of $z = f(a, b)$ and let $\mathbf{u}$ be a unit vector in the xy-plane. If we want to find the rate of change of f in the direction $\mathbf{u}$at $P_0(a, b)$, we can't simply use $f_x(a, b)$ or $f_y(a, b)$ unless $\mathbf{u} = \langle 1, 0\rangle$ or $\mathbf{u} = \langle 0, 1\rangle$, but it is a combination of the above.

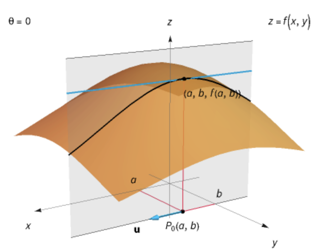


Figure 5.1: Sample $\mathbf{u}$ on a surface given by z

The derivative must be computed along a line l in the xy-plane that faces the same direction as $\mathbf{u}$. Now imagine Q, the plane perpendicular to the xy-plane containing l. This plane cuts into a surface $z = f(x, y)$ in a curve C. If we consider two points P_0 and P, then we can find the slope of the secant line between these two points:

$$\frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

The derivative of f in the direction of $\mathbf{u}$ is obtained by letting $h \to 0$; when this limit exists, it's called the directional derivative of f at (a, b) in the direction of $\mathbf{u}$.

Definition – Directional Derivative

Let f be differentiable at (a, b) and let $u = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of $\mathbf{u}$ is:

$$D_u f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

Theorem 15.10 – Directional Derivative

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of $\mathbf{u}$ is:

$$D_u f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle$$

2 The Gradient Vector

The vector $\langle f_x(a, b), f_y(a, b) \rangle$ that appears in the above dot product is important in it's own right; it is called the gradient of f.

Definition – Gradient (Two Dimensions)

Let f be differentiable at the point (x, y). The gradient of f at (x, y) is the vector-valued function:

$$\nabla f(x, y) = \langle f_x(a, b), f_y(a, b) \rangle = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

With the definition of the gradient, we can write the directional derivative of f at (a, b) in the direction of $\mathbf{u}$ as:

$$D_u f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$$

3 Interpretations of the Gradient

Using the properties of the dot product, we can see that:

$$\begin{aligned} D_u f(a, b) &= \nabla)(a, b) \cdot \mathbf{u} \\ &= |\nabla f(a, b)||\mathbf{u}| \cos\theta \\ &= |\nabla f(a, b)| \cos\theta \end{aligned}$$

Theorem 15.11 – Directions of Change

Let f be a differentiable function at (a, b) with $\nabla f(a, b) \neq 0$:

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of increase in this direction is $|\nabla f(a, b)|$.

2. f has its maximum rate of decrease at (a, b) in the direction $-\nabla f(a, b)$. The rate of increase in this direction is $-|\nabla f(a, b)|$.

3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$

4 The Gradient and Level Curves

Theorem 15.12 – The Gradient and Level Curves

Given a function f differentiable at (a, b), the tangent line to the level curve of f at (a,b) is orthogonal to the gradient $\nabla f(a, b)$ provided by $\nabla f(a, b) \neq 0$

5 The Gradient in Three Dimensions

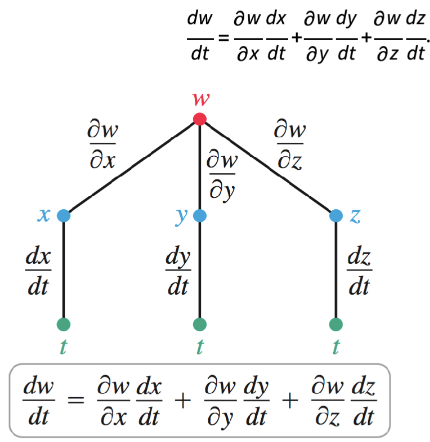


Figure 5.2: Visualized Gradient in Three Dimensions

Definition – Directional Derivative and Gradient in Three Dimensions

Let f be differentiable at (a, b, c) and let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a,b,c) in the direction of **u** is:

$$D_u f(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b, c)}{h}$$

The gradient of f at the point (x, y, z) is the vector valued function:

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= f_x(x, y, z)\ \hat{i} + f_y(x, y, z)\ \hat{j} + f_z(x, y, z)\ \hat{k} \end{aligned}$$

CHAPTER 6

TANGENT PLANES AND LINEAR APPROXIMATION

1 Tangent Planes

Recall that a surface in $\mathbb{R}^3$ may be defined in at least two different ways:

1. **Explicitly** in the form $z = f(x, y)$, or
2. **Implicitly** in the form $F(x, y, z) = 0$.

$$\frac{d}{dt}[F(x(t), y(t), z(t))] = \nabla F(x, y, z) \cdot \mathbf{r}'(t)$$

Definition – Equation of the Tangent Plane for $F(x, y, z) = 0$

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq 0$. The plane tangent to surface at P_0 is called the tangent plane, and it is the plane that passes through P_0 orthogonal to $\nabla F(a, b, c)$.

Important – Tangent Plane for $z = f(x, y)$

Let f be differentiable at that point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is:

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

2 Linear Approximation

A tangent line at a point gives a good approximations to the function's behaviour near that point. This method is called linear approximation.

Definition – Linear Approximation

Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation:

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

We can construct a formula for linear approximations in three variables the same way.

3 Differentials and Change

The exact change in the function between the points (a, b) and (x, y) is:

$$\Delta z = f(x, y) - f(a, b)$$

Replacing $f(x, y)$ by its linear approximation, the change Δz is approximated by:

$$\Delta z \approx L(x, y) - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Definition – The Differential dz

Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x + dx, y + dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(x, y)dx + f_y(x, y)dy$$

Part III

Small-Signal Analysis

CHAPTER 1

SMALL-SIGNAL ANALYSIS

$$z = f(x, y) = f(a, b) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=a \\ y=b}} (x - a) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a \\ y=b}} (y - b)$$

We need to introduce a definition for the Jacobian matrix before we can describe the small signal model procedure.

1 Jacobian Matrix

Express $F(x, y, z)$ in new variables as follows:

$$x = f(u, v, w) \tag{1.1}$$
$$y = g(u, v, w) \tag{1.2}$$
$$z = h(u, v, w) \tag{1.3}$$

Definition – Jacobian Matrices

A 3×3 Jacobian matrix $\mathbf{J}$ has the form:

$$J = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{bmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)} \tag{1.4}$$

2 Small-Signal Model for a Single Input and Single Output

Definition

The nomenclature Δu and $\hat{u}(t)$ are interchangeable. Usually we use the latter to denote small variations associated with time.

$$Y + Output(t) = g(U + Input(t))$$

3 Small-Signal Model for the Multivariable Case Given a System Model

We start with some definitions:

Definitions – state variable x, input variable u, and output variable y

State Variable x – variable in a first order derivative $\frac{dx}{dt}$
Input Variable u – variable representing an excitation signal $\frac{dx}{dt} = f(x, u)$
Output Variable y – Function of the State Variable x and input variable $u : y = g(x, u)$

Definition – State Space Equations

$$\frac{dx_1}{dt} = f_1(x_1(t), ..., x_n(t), u_1(t), ..., u_m(t))$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_2(x_1(t), ..., x_n(t), u_1(t), ..., u_m(t))$$

Definition – Output Equations

$$y_1 = g_1(x_1(t), ..., x_n(t), u_1(t), ..., u_m(t))$$
$$\vdots$$
$$y_k = g_k(x_1(t), ..., x_n(t), u_1(t), ..., u_m(t))$$

Solving DEs for the general case is beyond the scope of this course as it requires knowledge of discrete mathematics and numerical methods.

3.1 Observations on Linear and Non-Linear Systems of Equations

1. If the differential equation set is nonlinear in any of the state variables or input variables, then the differential equation set is nonlinear.

2. If an output equation is nonlinear in any of the state variables or input variables, then the output equation is nonlinear.

3. In the nonlinear case, the right-hand side of the differential equation set, or the right-hand side of the output equation cannot be expressed in a matrix form since one or more of the expressions is non-linear.

4 Small-Signal Modelling Procedure

1. Expand the differential state space equations and algebraic output equations and neglect all terms with derivatives greater than first order. The result is a set of coupled linear DEs and linear output equations.

2. Set all time derivatives and small-signal perturbation terms to zero, then:

   (a) Determine the equilibrium solution for the state variables given in the equilibrium values for the input variables.

   (b) Determine the equilibrium solution for the output variables.

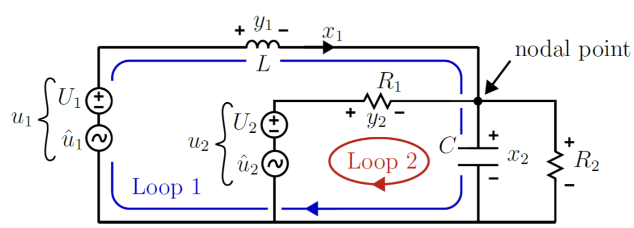
3. Determine the elements of the Jacobian matrices for the small-signal model. The elements of the Jacobian matrices are a function of the equilibrium values for the input variables and state variables.

An example of small-signal modelling will be provided in the following chapter.

CHAPTER 2

SMALL-SIGNAL ANALYSIS EXAMPLE

Problem: Write out the system equations and determine the small signal model for the circuit given below.



Begin by assigning $u_1(t) \rightarrow v_1(t)$, $u_2(t) \rightarrow v_2(t)$, $y_1(t) \rightarrow v_L(t)$, $y_2(t) \rightarrow v_{R_2}(t)$, $x_1(t) \rightarrow v_L(t)$, $x_1(t) \rightarrow v_C(t)$.

Step 1: Write out differential equations and output equations for the circuit. Differential (obtained by applying KVL to both loops):

$$y_1 = v_L(t) = L\frac{dx_1}{dt} = u_1 - x_2, y_2 = R_1(C\frac{dx_2}{dt} + \frac{x_2}{R_2} - x) = u_2 - x_2$$

Output equations:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.1}$$

Step 2: Convert the differential equations into state space form.

$$\frac{dx_1}{dt} = \frac{u_1 - x_2}{L}, \frac{dx_2}{dt} = \frac{1}{R_1C}u_2 + \frac{1}{C}x_1 - \left(\frac{1}{R_2C} + \frac{1}{R_1C}\right)x_2$$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{C} & -\frac{1}{R_2C} - \frac{1}{R_1C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{R_1C} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.2}$$

Step 3: Set above equal to [0 0] and solve for $[x_1 \; x_2]$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_2} + \frac{1}{R_1} & -\frac{1}{R_1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

$$\begin{cases} x_1 = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)u_1 - \frac{1}{R_1}u_2 \\ x_2 = u_1 \end{cases} \tag{2.3}$$

Substitute back into $[y_1\ y_2] \rightarrow y_1 = 0, y_2 = u_2 - u_1$.

Step 4: Apply the small signal model. First do state equations:

$$\frac{d\hat{x}_1}{dt} = f_1(X_1, X_2, U_1, U_2) + \left.\frac{\partial f_1(X_1, X_2, U_1, U_2)}{\partial u_1}\right|_{u_1=U_1} \hat{u}_1 + \left.\frac{\partial f_1(X_1, X_2, U_1, U_2)}{\partial u_2}\right|_{u_2=U_2} \hat{u}_2 + \left.\frac{\partial f_1(X_1, X_2, U_1, U_2)}{\partial x_1}\right|_{x_1=X_1} \hat{x}_1 + \left.\frac{\partial f_1(X_1, X_2, U_1, U_2)}{\partial x_2}\right|_{x_2=X_2} \hat{x}_2$$

$$\frac{d\hat{x}_2}{dt} = f_2(X_1, X_2, U_1, U_2) + \left.\frac{\partial f_2(X_1, X_2, U_1, U_2)}{\partial u_1}\right|_{u_1=U_1} \hat{u}_1 + \left.\frac{\partial f_2(X_1, X_2, U_1, U_2)}{\partial u_2}\right|_{u_2=U_2} \hat{u}_2 + \left.\frac{\partial f_2(X_1, X_2, U_1, U_2)}{\partial x_1}\right|_{x_1=X_1} \hat{x}_1 + \left.\frac{\partial f_2(X_1, X_2, U_1, U_2)}{\partial x_2}\right|_{x_2=X_2} \hat{x}_2$$

Now do output equations:

$$Y_1 + \hat{y} =_1= g_1(X_1, X_2, U_1, U_2) + \left.\frac{\partial g_1(X_1, X_2, U_1, U_2)}{\partial u_1}\right|_{u_1=U_1} \hat{u}_1 + \left.\frac{\partial g_1(X_1, X_2, U_1, U_2)}{\partial u_2}\right|_{u_2=U_2} \hat{u}_2 + \left.\frac{\partial g_1(X_1, X_2, U_1, U_2)}{\partial x_1}\right|_{x_1=X_1} \hat{x}_1 + \left.\frac{\partial g_1(X_1, X_2, U_1, U_2)}{\partial x_2}\right|_{x_2=X_2} \hat{x}_2$$

$$Y_2 + \hat{y} =_2= g_1(X_1, X_2, U_1, U_2) + \left.\frac{\partial g_2(X_1, X_2, U_1, U_2)}{\partial u_1}\right|_{u_1=U_1} \hat{u}_1 + \left.\frac{\partial g_2(X_1, X_2, U_1, U_2)}{\partial u_2}\right|_{u_2=U_2} \hat{u}_2 + \left.\frac{\partial g_2(X_1, X_2, U_1, U_2)}{\partial x_1}\right|_{x_1=X_1} \hat{x}_1 + \left.\frac{\partial g_2(X_1, X_2, U_1, U_2)}{\partial x_2}\right|_{x_2=X_2} \hat{x}_2$$

Put all equations in matrix form – start with the state equations:

$$\begin{bmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{bmatrix} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial u_1}\right|_{u_1=U_1} & \left.\frac{\partial f_1}{\partial u_2}\right|_{u_2=U_2} \\ \left.\frac{\partial f_2}{\partial u_1}\right|_{u_1=U_1} & \left.\frac{\partial f_2}{\partial u_2}\right|_{u_2=U_2} \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} + \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{x_1=X_1} & \left.\frac{\partial f_1}{\partial x_2}\right|_{x_2=X_2} \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{x_1=X_1} & \left.\frac{\partial f_2}{\partial x_2}\right|_{x_2=X_2} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{2.4}$$

Remember this all equals [0 0].

$$\begin{bmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{bmatrix} = \mathbb{J}_f \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \mathbb{B}_f \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} \tag{2.5}$$

$$\mathbb{J}_f = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -(\frac{1}{R_2C} + \frac{1}{R_1C}) \end{bmatrix}, \mathbb{B}_f = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{R_1C} \end{bmatrix}$$

Put output equations in matrix form:

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \mathbb{J}_g \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \mathbb{B}_g \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} \tag{2.6}$$

where:

$$\mathbb{J}_g = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} \text{ and } \mathbb{B}_g = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

Solve for the circuit:

$$\mathbb{J}_g = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \text{ and } \mathbb{B}_g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Part IV

Multiple Integration

CHAPTER 1

DOUBLE INTEGRALS OVER RECTANGULAR REGIONS

1 Volumes of Solids

We assume $z = f(x, y)$ is a non-negative continuous function on a rectangular region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. A partition of R is formed by dividing R into n rectangular regions using lines running parallel to the x and y axes. Essentially, we have $\Delta A_k = \Delta x_k \Delta y_k$.

Therefore, the volume of the k-th box:

$$f(x_k^*, y_k^*)\Delta A_k = f(x_k^*, y_k^*)\Delta x_k \Delta y_k$$

The sum of the volumes of the n boxes gives an approximation to the volume of the solid:

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

Definition – Double Integrals

A function f defined on a rectangular region R in the xy-place is integrable if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ exists for all partitions of R. The limit is the double integral of f over R, denoted as:

$$\iint f(x, y)dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

2 Iterated Integrals

$$
\begin{aligned}
V &= \iint f(x,y)dA = \iint (6-2x-y)dA \\
V &= \int_0^1 A(x)dx \\
A(x) &= \int_0^2 (6-2x-y)dy \\
V &= \int_0^1 A(x)dx = \int_0^1 (\int_0^2 (6-2x-y)dy)dx
\end{aligned}
$$

The expression that appears on the right side of this equation is called an **iterated integral**.

Theorem 16.1 – (Fubini) Double Integrals over Rectangular Regions

Let f be continuous on a rectangular region. The double integral of f over R may be evaluated by either of two integrated integrals:

$$\iint f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

The importance of Fubini's Theorem is that it says double integrals may be evaluated using iterated integrals, and also that the order of integration doesn't matter.

3 Average Value

Recall the average value of the integrable function f over the interval $[a,b]$ is:

$$\bar{g} = \frac{1}{b-a}\int_a^b f(x)dx$$

To find the average value of an integrable function f over a region R, we integrate f over R and divide the result by the "size" of R, which is the area of R in the two-variable case.

Definition – Average Value of a Function Over a Plane Region

The average value of an integrable function over f over a region R is:

$$\bar{f} = \frac{1}{\text{area of } R}\iint f(x,y)dA$$

CHAPTER 2

DOUBLE INTEGRALS OVER GENERAL REGIONS

1 Iterated Integrals

Double integrals over non-rectangular regions are also evaluated using iterated integrals, but in this case, the order of integration is critical.

Step-by-step procedure for iterated integrals:

1. Convert to an iterated integral,
2. Evaluate inner integral with respect to y,
3. Simplify,
4. Evaluate outer integral with respect to x.

$$\begin{aligned}
\iint 2x^2y\ dA &= \int_{-2}^{2}\int_{3x^2}^{16-x^2} 2x^2y\ dy\ dx \\
&= \int_{-2}^{2} x^2y^2\Big|_{3x^2}^{16-x^2} dx \\
&= \int_{-2}^{2} x^2((16-x^2)^2-(3x^2)^2)dx \\
&= \int_{-2}^{2} (-8x^6-32x^4+256x^2)dx \\
&\approx 663.2
\end{aligned}$$

Theorem 16.2 – Double Integrals over Non-Rectangular Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on

R, then:

$$\iint f(x,y)dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\ dy\ dx$$

If R is instead a region bounded by the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$, then:

$$\iint f(x,y)dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\ dx\ dy$$

2 Choosing and Changing the Order of Integration

Just examples.

3 Regions Between Two Surfaces

Volume of the solid between the surfaces is:

$$V = \iint (f(x,y) - g(x,y))dA$$

4 Decomposition of Regions

By partitioning regions and using Riemann sums, it can be shown that:

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

CHAPTER 3

DOUBLE INTEGRALS IN POLAR COORDINATES

1 Moving from Rectangular to Polar Coordinates

In polar coordinates, a polar rectangle has the form $R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$. The volume of the solid region beneath the surface $z = f(x, y)$ with a base R is approximately:

$$V = \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

This approximation to the volume is a Riemann sum. We let Δ be the maximum value of Δr and $\Delta\theta$. If f is continuous on R, then as $n \to \infty$ and $\Delta \to 0$, the sum approaches a double integral:

$$\iint f(x, y)dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{3.1}$$

Theorem 16.3 – Change of Variables for Double Integrals over Polar Rectangle Regions

Let f be continuous on the region R in the xy plane expressed in polar coordinates $R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha \leq= 2\pi$. Then f is integrable over R, and the double integral of f over R:

$$\iint f(x, y)dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta)r dr d\theta$$

2 More General Polar Regions

Theorem 16.4 – Change of Variables for Double Integrals over More General Polar Regions

Let f be continuous on the region R in the xy-plane expressed in polar coordinate as:

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$$

where $0 < \beta - \alpha \leq 2\pi$. Then:

$$\iint f(x, y) dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

3 Areas of Regions

Areas of Polar Regions

The are of the polar region $R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$, is:

$$A = \iint dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} r dr d\theta$$

CHAPTER 4

TRIPLE INTEGRALS IN RECTANGULAR COORDINATES

1 Triple Integrals in Rectangular Coordinates

Consider a function $w = f(x, y, z)$ that is defined on a closed and bounded region D of $\mathbb{R}^3$; despite the difficulty in representing it, we can still define the integral of f over D. We partition D into boxes that are wholly contained within D, with the kth box having $\Delta V_k = \Delta x_k, \Delta y_k, \Delta z_k$.

$$\iiint_D f(x, y, z) = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k \tag{4.1}$$

Notice the analogy between double and triple integrals:

- Area of $R = \int\int_R dA$, and
- Volume of $D = \int\int\int_D dV$

2 Finding Limits of Integration

We can write the triple integral as an iterated integral.

$$\iiint_D f(x, y, z)dV = \iint_R \left(\int_G^H f(x, y, z)dz \right) dA \tag{4.2}$$

Theorem – Triple Integrals

To integrate over all points of D, we carry out the following:

1. Integrate with respect to z from $z = G(x, y)$ to $z = H(x, y)$; the result is generally a function of x and y.
2. Integrate with respect to y from $y = g(x)$ to $y = h(x)$; the result is generally a function of x.

3. Integrate with respect to x from $x = a$ to $x = b$; the result is always a number that doesn't depend on x, y, or z.

Note: this Theorem is a version of Fubini's Theorem.

3 Changing the Order of Integration

We can simplify the solving of a triple integral by choosing an appropriate order of integration. Often we don't know whether a particular order will work, some trial and error is required.

4 Average Value of a Function of Three Variables

Definition – Average Value of a Function of Three Variables

If f is continuous on a region D of $\mathbb{R}^3$, then the average value of f over D is:

$$\bar{f} = \frac{1}{\text{volume of} D} \iiint_D f(x, y, z) dV \tag{4.3}$$

CHAPTER 5

TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

1 Cylindrical Coordinates

Transformations Between Cylindrical and Rectangular Coordinates

- Rectangular $\rightarrow$ Cylindrical
  - $r^2 = x^2 + y^2$
  - $\tan\theta = y/x$
  - $z = z$
- Cylindrical $\rightarrow$ Rectangular
  - $x = r\cos\theta$
  - $y = r\sin\theta$
  - $z = z$

2 Integration in Cylindrical Coordinates

Theorem 16.6 – Change of Variables for Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region D, expressed in cylindrical coordinates as:

$$D = \{(r, \theta, z) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta\beta, G(x, y) \leq z \leq H(x, y)\}$$

Then, f is integrable over D, and the triple integral of f over D:

$$\iiint f(x, y, z)dV = \int_{\alpha}^{\beta} \int_{g}^{h} \int_{G}^{H} f(r\cos\theta, r\sin\theta, z)dzrdrd\theta$$

3 Spherical Coordinates

- ρ is the distance from the origin to P.
- ϕ is the angle between the positive z-axis and the line OP.
- θ is the same angle as in cylindrical coordinates; it measures rotation about the z-axis relative to the positive x-axis.

4 Integration in Spherical Coordinates

Theorem 16.7 – Change of Variables for Triple Integrals in Spherical Coordinates

Let f be continuous over the region D, expressed in spherical coordinates as:

$$D = \{(\rho, \phi, \theta) : 0 \leq g(\phi, \theta) \leq \rho \leq h(\phi, \theta), a \leq \phi \leq b, \alpha\theta\beta\}$$

Then, f if integrable over D, and the triple integral of f over D is:

$$\iiint f(x, y, z)dV = \int_{\alpha}^{\beta} \int_{a}^{b} \int_{g}^{h} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi d\rho d\phi d\theta$$

CHAPTER 6

CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

1 Recap of Change of Variables

We use the change of variables strategy to help simplify a single-variable integral. For example, to simplify the integral $\int_0^1 s\sqrt{2x+1}dx$, we choose a $u = 2x+1$ and $du = 2dx$. Therefore, we have:

$$\int_0^1 2\sqrt{2x+1}dx = \int_1^3 \sqrt{u}du$$

2 Transformations in the Plane

A change of variables in a double integral is a transformation that relates a pair of variables to another. $(x, y) = T(u, v)$ is compactly written as:

$$T : x = g(u, v) \text{ and } y = h(u, v)$$

One-to-One Transformation

A transformation T from a region S to a region R is one-to-one on S if $T(P) = T(Q)$ only when $P = Q$.

Jacobian Determinant of a Transformation of Two Variables

Given a transformation $T : x = g(u, v), y = h(u, v)$ where g and h are differentiable on a region of the uv-plane, the Jacobian determinant of T is:

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{6.1}$$

Theorem 16.8 – Change of Variables for Double Integrals

Let T be a transformation that maps a closed bounded region S in the uv-plane to a region R in the xy-plane. Assume T is injective on the interior of S and g, h have continuous first partial derivatives there. If f is continuous on R, then:

$$\iint_R f(x,y)dA = \iint_S f(g(u,v)h(u,v))|J(u,v)|dA \tag{6.2}$$

3 Change of Variables in Triple Integrals

Definition – Jacobian Determinant of a Transformation of Three Variables

Given a transformation T where g, h, p are differentiable on a region of uvw-space, the Jacobian is given by:

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \tag{6.3}$$

Theorem 16.9 – Change of Variables for Triple Integrals

Let $T(u,v,w)$ be a transformation that maps a closed bounded region S to a region $D = T(S)$. Assume T is one-to-one on the interior of S and g, h, p have continuous first partial derivatives there. If f is continuous on D, then:

$$\iint_D f(x,y,z)dV = \iint_S f(g(u,v,w), h(u,v,w), p(u,v,w))|J(u,v,w)|dV \tag{6.4}$$

4 Strategies for Choosing New Variables

Here are some suggestions for finding new variables of integration. These apply to both double and triple integrals.

1. **Aim for simple regions of integration in the uv plane.** The new region should be as simple as possible. For e.g., double integrals are easiest to evaluate over rectangular regions with sides parallel to the coordinate axes.

2. **Is $(x,y) \to (u,v)$ or $(u,v) \to (x,y)$ better?** Depending on the problem, inverting the transformation may be easy, difficult, or impossible.

3. **Let the integrand suggest new variables.**

4. **Let the region suggest new variables.**

Part V

Vector Calculus

CHAPTER 1

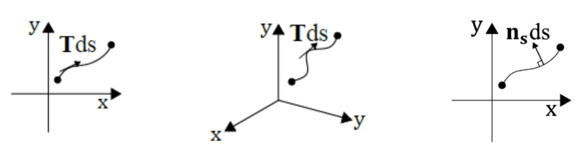
LINE INTEGRALS OF VECTOR DOT PRODUCTS

1 Geometrical Properties

We will describe in pictorial form the types of geometries to consider when performing integrals over curved lines and curved surfaces. We will consider 2D problems as well as 3D problems.

1. $\mathbf{T}$ – unit vector tangent to the contour
2. $\mathbf{n}_s$ – unit vector normal to the contour
3. ds – differential arc length
4. $\mathbf{T}ds = \mathbf{ds}$ – differential arc length

Open Contour Geometries



Tangent vector in 2d (open contour) | Tangent vector in 3d (open contour) | Normal vector in 2d (open contour)

Figure 1.1: Open Contour Geometries

Closed Contour Geometries

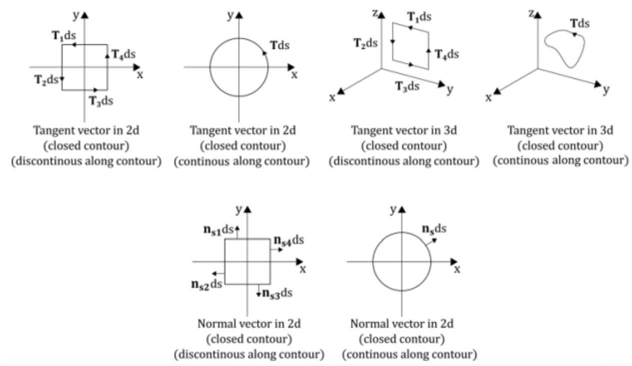


Figure 1.2: Closed Contour Geometries

Closed and Open Surfaces

It is obvious to see that any closed surface can be decomposed into a group of open surfaces.

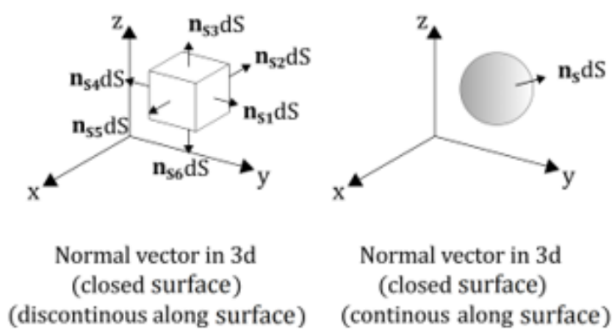


Figure 1.3: Orientation of a Differential Surface Area Vector

1. $\mathbf{n}_S$ – unit vector normal to a differential surface area
2. dS – magnitude of differential surface area
3. $\mathbf{n}_S dS = \mathbf{dS}$ – differential surface area

Flux and Circulation

Flux means something that leaves or enters a 2D contour in a plane or a surface in a 3D domain.

Circulation is a measure of how much rotation exists in a vector field. Both of these values are scalars.

2 Integral Definitions

2.1 Contour Integrals

1. Integrating the scalar density along the length of an open contour

$$\int_C \lambda(s) ds \tag{1.1}$$

2. Integrating the scalar density along the length of a closed contour

$$\oint_C \lambda(s) ds \tag{1.2}$$

3. Average of f over an open contour

$$f_{avg} = \frac{\int_C f ds}{\int_C ds} \tag{1.3}$$

4. Average of f over a closed contour

$$f_{avg} = \frac{\oint_C f ds}{\oint_c ds} \tag{1.4}$$

5. Flux from an open contour

$$\int_C \mathbf{F} \cdot \mathbf{n}_s ds \tag{1.5}$$

6. Flux from a closed contour

$$\oint_C \mathbf{F} \cdot \mathbf{n}_s ds \tag{1.6}$$

7. Circulation of an open contour

$$\int_C \mathbf{F} \cdot \mathbf{T} ds \tag{1.7}$$

8. Circulation of a closed contour

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds \tag{1.8}$$

2.2 Surface Area Integrals

1. Integrating the scalar density over an open surface

$$\iint_S \sigma(S) dS \tag{1.9}$$

2. Integrating the scalar density over a closed surface

$$\oiint_S \sigma(S) dS \tag{1.10}$$

3. Average of f over an open surface

$$f_{avg} = \frac{\iint_S f dS}{\iint_S dS} \tag{1.11}$$

4. Average of f over a closed surface

$$f_{avg} = \frac{\oiint_S f dS}{\oiint_S dS} \tag{1.12}$$

5. Flux from an open surface

$$\iint_S \mathbf{F} \cdot \mathbf{n}_s dS \tag{1.13}$$

6. Flux from a closed surface

$$\oiint_S \mathbf{F} \cdot \mathbf{n}_s dS \tag{1.14}$$

CHAPTER 2

SCALAR FUNCTION AND FLUX WITH PARAMETRIC REPRESENTATION

1 Surface Parameterization

First step is to map the surface to a plane, there are two ways of doing so:

Parameterized Representation

We can make the mapping $(x, y, z) = (x(u, v), y(u, v), f(x(u, v), y(u, v)))$ on the surface S.

Explicit Representation

Points of the xy plane have coordinates (x, y), so we can map the point (x, y) to $(x, y, z) = (x, y, f(x, y))$.

2 Surface Integrals of Scalar-Values Functions

We now develop the surface integral of a function f on a smooth parameterized surface S described by:

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle \tag{2.1}$$

We then need to compute dS by finding the two special vectors tangent to the surface at P.

- $\vec{t_u}$ is a vector tangent to the surface corresponding to a change in u with v held constant, and
- $\vec{t})_v$ is a vector tangent to the surface corresponding to a change in v with u held constant.

$$\vec{t_u} = \frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \tag{2.2}$$

$$\vec{t_v} = \frac{\partial \vec{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \tag{2.3}$$

We then apply the cross product to find the area of the parallelogram:

$$|\vec{t_u}\Delta u \times \vec{t_v}\Delta v| = |\vec{t_u} \times \vec{t_v}|\Delta u\Delta v = \Delta S_k = dS \tag{2.4}$$

3 Computing Flux

Definition – Flux Integral (Parameterization)

$$\iint_S \mathbf{F} \cdot \mathbf{n}_S dS = \iint_R \langle g(u,v), h(u,v), p(u,v)\rangle \cdot \mathbf{t}_u \times \mathbf{t}_v dudv \tag{2.5}$$

CHAPTER 3

CIRCULATION AND FLUX, DIVERGENCE AND CURL OPERATORS

Any scalar function $f(x)$ can be decomposed into a function with odd and even characteristics.

$$f_{odd}(x) = \frac{1}{2}(f(x) - f(-x))$$
$$f_{even}(x) = \frac{1}{2}(f(x) + f(-x))$$
$$f(x) = f_{odd}(x) + f_{even}(x)$$

1 Vector Decomposition Description

Imagine we have a 2d vector field $\mathbf{F} = \langle x - y, y + x \rangle$. We can write this vector out as a sum of two vector fields which we claim have different properties.

$$\mathbf{F}_{flux} = \langle x, y \rangle \tag{3.1}$$
$$\mathbf{F}_{circ} = \langle -y, x \rangle \tag{3.2}$$
$$\mathbf{F} = \mathbf{F}_{flux} + \mathbf{F}_{circ} \tag{3.3}$$

1.1 Description of $\mathbf{F}_{flux}$

The field $\mathbf{F}_{flux}$ is a radial flux density that originates from a single point, so the point can be viewed as a source or sink. The flux integral can be linked to the strength of the source.

We propose the divergence of a vector field as the strength of the source, which is the flux integral divided by the area enclosed by the contour.

1.2 Description of $\mathbf{F}_{circ}$

The field $\mathbf{F}_{circ}$ is a circulation density whose field lines form closed contours that are centered at the origin. We can characterize the field pattern using the following minimum set of properties:

- The field lines encircle an axis which we call an axis of rotation.

- The circulation integral divided by the area enclosed by the contour can be linked to the strength of rotation.

We also need to define a pseudovector as the axis of rotation does not fir the description of a standard vector.

We define the curl of the vector field $\mathbf{F}$ by the product of the strength of rotation, and the unit pseudovector representing the direction of the axis of rotation.

2 Intuitive Understanding of the Divergence and Curl of a Vector Field

- Compute the flux and circulation over the closed contour for the vectors $\mathbf{F}$, $\mathbf{F}_{flux}$, and $\mathbf{F}_{circ}$.
- Normalize the flux and circulation integrals. In this case, we divide by the area enclosed by the contour so that the final result does not depend on the radius of the contour.
- Show the results in the table form and analyze them.

We will consider an example: the contour for a circle of radius r centered at the origin is parameterized as:

$$C : \langle r\cos(t), r\sin(t)\rangle \qquad 0 \leq t \leq 2\pi$$

	Flux	Flux/Area	Circulation	|Circulation/Area|
	$\oint_C \mathbf{F}\cdot\mathbf{n}_s ds$	$\dfrac{\oint_C \mathbf{F}\cdot\mathbf{n}_s ds}{\pi r^2}$	$\oint_C \mathbf{F}\cdot\mathbf{T} ds$	$\dfrac{\oint_C \mathbf{F}\cdot\mathbf{T} ds}{\pi r^2}$
$\mathbf{F}_{circ}$	0	0	$2\pi r^2$	2
$\mathbf{F}_{flux}$	$2\pi r^2$	2	0	0
$\mathbf{F}$	$2\pi r^2$	2	$2\pi r^2$	2

Table 3.1: Flux, Circulation, and Normalized Vector Values for the Vectors $\mathbf{F}$, $\mathbf{F}_{flux}$, and $\mathbf{F}_{circ}$

Some observations of note:

1. Normalizing the circulation and flux integrals for a simple linear problem allowed us to remove the dependency on the shape and size of the contour.
2. The magnitude of the curl of a vector field is attributed to a rotational field $\mathbf{F}_{circ}$ and the divergence of a vector field is attributed to a source field $\mathbf{F}_{flux}$.
3. The flux integral acts as a filter that only selects $\mathbf{F}_{flux}$ and likewise for $\mathbf{F}_{circ}$.
4. These concepts can easily be generalized to 3d cases.

3 Formal Definitions of the Divergence and Curl of a Vector Field

Definition – Divergence of a Vector Field F

$$\lim_{A \to 0} \frac{\oint_C \mathbf{F} \cdot \mathbf{n}_s ds}{V} \tag{3.4}$$

$$\mathbf{F} = \langle f(x, y)g(x, y), h(x, y)\rangle = \langle F_x, F_y, F_z\rangle \tag{3.5}$$

Definition – Curl of a Vector Field F in 2d

$$\lim_{A_{\hat{z}} \to 0} \frac{\oint_C \mathbf{F} \cdot \mathbf{T} ds}{A_{\hat{z}}} \hat{z} \tag{3.6}$$

$$\mathbf{F} = \langle f(x, y), g(x, y), 0)\rangle = \langle F_x, F_y, 0\rangle = \langle F_x, F_y\rangle \tag{3.7}$$

4 Concluding Observations

4.1 Physical Meaning of $\nabla \cdot \mathbf{F}$ (a scalar)

We mentioned earlier that the divergence of a vector field indicates the strength of a source at a particular point. Now assume that we give the source a physical meaning, like mass or charge density.
If we know the field and sweep over a region computing at each point $\nabla \cdot \mathbf{F}$, we are actually filtering out information telling us where charge exists and the magnitude of the charge density at any given location.

4.2 Physical Meaning of $\nabla \times \mathbf{F}$ (a pseudovector)

Hence, $\nabla \times \mathbf{F} = \mathbf{J}$.If we give $\mathbf{J}$ a physical meaning such as a current flux density, then it implies that the field $\mathbf{F}$ represents a circulation density. If we know the field $\mathbf{F}$ and we sweep over the the region while computing $\nabla \times \mathbf{F}$ at all points, then we are filtering out information telling us where current flux density exists and what its magnitude and direction is at any given location.

5 Vector Operators

Vector Operation	Expression
Gradient $\nabla(\phi(x,y,z))$	$\nabla(\phi) = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$
Curl $\nabla \times \mathbf{F}$	$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x}, \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y}, \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$
Divergence $\nabla \cdot \mathbf{F}$	$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

Table 3.2: A Summary of the Primitive Vector Operators

CHAPTER 4

HEMHOLTZ DECOMPOSITION, DIVERGENCE AND STOKES' THEOREM

1 Hemholtz' Decomposition Theorem

$$\mathbf{F} = \mathbf{F}_{flux} + \mathbf{F}_{circ} = -\nabla\phi + \nabla\times\mathbf{A} \tag{4.1}$$

$\mathbf{A}$ is defined as the vector potential and ϕ as the scalar potential.

2 Divergence and Stokes' Theorem

THe Divergence Theorem relates a volume integral to a closed surface integral whereas Stokes' Theorem relates an open surface integral to a closed contour integral.

Divergence Theorem in 3d

$$\oint\!\!\oint_S \mathbf{F}\cdot\mathbf{n}_S dS = \iiint_V \nabla\cdot\mathbf{F} dV \tag{4.2}$$

$\mathbf{n}_s$ is the unit vector normal to a differential surface area element on the closed surface S. It points outwards from the surface S that encloses the volume V.

Stokes' Theorem in 3d

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \iint_{S_1} \nabla\times\mathbf{F}\cdot\mathbf{n}_{S_1}dS_1 = \iint_{S_2} \nabla\times\mathbf{F}\cdot\mathbf{n}_{S_2}dS_2 \tag{4.3}$$

CHAPTER 5

COMPUTING A VECTOR FIELD WHEN $\nabla \times \vec{F}$ & $\nabla \cdot \vec{F}$ ARE GIVEN

Many problems in nature (electrostatics, magnetostatics, heat transfer, fluid flow, etc.) are described by either of the following equations:

$$\nabla \cdot \vec{F} = \rho(x,y,z) \qquad \nabla \times \vec{F} = 0 \text{ irrotational field} \tag{5.1}$$

$$\nabla \cdot \vec{F} = 0 \qquad \nabla \times \vec{F} = \vec{J}(x,y,z) \text{ source-free field} \tag{5.2}$$

1 Symmetries

There are various types of symmetries. The main criterion is that the vector $\vec{F}$ is aligned with one of the basis vectors in the appropriate coordinate system.

1.1 Planar Symmetry

Consider $\vec{F} = f(x)\hat{x}$ in 3d Cartesian coordinates. The expression indicates that the vector points in the $\hat{x}$ direction and the magnitude depends only on the x position.

1.2 Cylindrical Symmetry

Let $\vec{F} = f(\sqrt{x^2+y^2})(-\frac{y}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y}) = f(r)\hat{\theta}$. The vector points in the $\hat{\theta}$ direction and its magnitude depends only on position r where $r = \sqrt{x^2+y^2}$.

1.3 Cylindrical Symmetry

Let $\vec{F}$ be the vector such that $\vec{F} = f(\sqrt{x^2+y^2})(\frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y}) = f(r)\hat{r}$. The vector points in the $\hat{r}$ direction and its magnitude depends only on position r where $r = \sqrt{x^2+y^2}$.

1.4 Spherical Symmetry

Let $\vec{F}$ be the vector $\vec{F} = f(\sqrt{x^2+y^2+z^2})(\frac{y}{\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{y} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{z}) = f(r)\hat{r}$. The vector points in the direction of $\hat{r}$ and its magnitude depends only on the position r where $r = \sqrt{x^2+y^2+z^2}$.

2 Cases That Can be Solved Without Specialized Tools or Knowledge

1. An irrotational field $\vec{F}$ and symmetry $\rho(x, y, z)$:

$$\nabla \cdot \vec{F} = \rho(x, y, z) \tag{5.3}$$
$$\nabla \times \vec{F} = 0 \tag{5.4}$$

We compute $\vec{F}$ using the Divergence Theorem:

$$\oiint_S d\vec{S} = \iiint_V \nabla \cdot \vec{F} dV = \iint_V \rho dV \tag{5.5}$$

2. A source free field $\vec{F}$, and symmetry for $\vec{J}(x, y, z)$:

$$\nabla \cdot \vec{F} = 0 \tag{5.6}$$
$$\nabla \times \vec{F} = \vec{J}(x, y, z) \tag{5.7}$$

$\vec{F}$ is computed using Stokes' Theorem:

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_S \nabla \times \vec{F} \cdot d\vec{s} = \iint_S \vec{J} \cdot d\vec{s} \tag{5.8}$$

3 Summary of Conditions for Symmetry

We invoke symmetry on equations (3) and (4) if ρ is a function of only variable in an appropriately chosen coordinate system. Likewise, we can only invoke symmetry on (6) and (7) if $\vec{J}$ is directed along one of the basis vectors in an appropriate coordinate system and has a magnitude that is a function of only one coordinate system.

CHAPTER 6

MODELLING USING DIRAC-DISTRIBUTION FUNCTIONS

1 Properties of Scalar Densities and Flux Densities

- ρ is a scalar density ont he right-hand side of the Divergence Theorem

$$\oiint_S \vec{F} \cdot \vec{dS} = \iiint_V \nabla \cdot \vec{F} dV = \iiint_V \rho dV$$

- $\vec{J}$ is a flux density on the right-hand side of Stokes' Theorem

$$\oint_C \vec{F} \cdot \vec{dS} = \iiint_S \nabla \times \vec{F} \cdot \vec{dS} = \iint_S \vec{J} \cdot \vec{dS}$$

2 Distribution Functions

Consider cases in which a scalar function or vector valued function is infinite or undefined at specific points and is zero elsewhere. We need to invoke distribution functions, otherwise known as Dirac-Delta functions. We use the symbol $\sigma(x)$ to represent a 1d Dirac-Delta function in a Cartesian coordinate system.

3 Gaussian Surfaces

The closed surface S associated with the left-hand side of the Divergence Theorem

$$\oiint_S \vec{F} \cdot \vec{dS} = \iiint_V \nabla \cdot \vec{F} dV = \iiint_V \rho dV \tag{6.1}$$

is commonly referred to as the Gaussian surface.

CHAPTER 7

SUPERPOSITION

1 Superposition Principle

$$\nabla \cdot [\vec{F}_1 + \vec{F}_2] = \nabla \cdot \vec{F}_1 + \nabla \cdot \vec{F}_2 = \rho_1 + \rho_2 \tag{7.1}$$

$$\nabla \times [\vec{F}_1 + \vec{F}_2] = \nabla \times \vec{F}_1 + \nabla \times \vec{F}_2 = \vec{J}_1 + \vec{J}_2 \tag{7.2}$$

2 Applying the Principles of Superposition

- A volume charge distribution has spherical symmetry if the charge depends only on the distance from a point in space and not direction.
- A volume charge distribution has cylindrical symmetry if the charge density depends only on the distance measured in a direction orthogonal to the cylinder's axis and not the position along the axis at which the measurement was made.
- A volume charge distribution has planar symmetry if the charge density depends only on the distance measured in a direction orthogonal to the plane and not the position on the plane at which the measurement was made.
- A surface current flux density distribution has cylindrical symmetry if the surface current flux density depends only on the distance measured in a direction orthogonal to the cylinder's axis. The surface current flux density can be distributed in two different ways: in a circumferential direction with respect to the cylinder's axis, or parallel to the cylinder's axis.
- A surface current flux density distribution has toroidal symmetry if the surface current flux density forms a closed loop parallel to the exterior portion of a donut-shaped object, and if the direction of the density if parallel to the plane defined by the toroid's axis and the direction radial to toroid's axis.
- A volumetric current flux density has planar symmetry if the current flux density depends only on the distance measured in a direction orthogonal to the plane and is oriented in a direction parallel to the plane.