MAT290: Advanced Engineering Mathematics

Student Textbook & Lecture Notes

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PREFACE

These notes have been collected from those I took from the course textbook and instructor-provided readings throughout the semester. These notes are not a full representation of the course's content, and **there may be information that I noted down incorrectly**. I am posting these notes to my website so that future MAT290 students can take whatever they need from it. The content in these notes was taken from: *Advanced Engineering Mathematics*, Seventh Edition, Dennis G. Zill.

I also have a GitHub repository containing all these notes on a chapter-by-chapter basis sorted by week. It can be found at `github.com/arnav-patil-12/mat290-notes`

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Part I

Complex Variables

CHAPTER 1

POWERS AND ROOTS

1 Polar Form

A nonzero complex number $z = x + yi$ can be written as $z = (r\cos\theta) + i(r\sin\theta)$. We call this the polar form of the complex number z. We call r the modulus, or length of z, denoted as $|z|$. The angle θ is the argument of z and is denoted as $\theta = \arg z$. Remember that $\theta \pm 2k\pi$ where $k \in \mathbb{Z}$ are all arguments of z.

The **principal argument** of z is in the interval $(-\pi, \pi]$ and is denoted by Arg (z).

2 Multiplication and Division

Multiplying two complex numbers is given by:

$$z_1 z_2 = r_1 r_2 e^{\theta_1 + \theta_2}$$

Division of two complex numbers is given by:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{\theta_1 - \theta_2}$$

Proofs of the above are elementary in that they were covered in kindergarten.

3 Integer Powers of z

The n^{th} power of a given complex number z is given by:

$$z^n = r^n e^{n\theta i}$$

4 De Moivre's Formula

When $z = \cos\theta + i\sin\theta$, we have $|z| = 1$, which yields:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

CHAPTER 2

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1 Exponential Functions

From Calculus II we already have defined a complex exponential function.

$$e^z = e^{x+yi} = e^x(\cos y + i \sin y)$$

Note that when $y = 0$, the function reduces to e^x.

2 Properties

We will begin by finding the derivative of e^z:

$$f'(z) = e^x \cos y + i(e^x \sin y) = f(z)$$

Therefore, we have that $\dfrac{d}{dz}e^z = e^z$.

3 Periodicity

The complex function $f(z) = e^z$ is periodic with period $2\pi i$. The strip $-\pi < y \leq \pi$ is called the **fundamental region** of the exponential function.

4 Polar Form of a Complex Number

As we saw in **Section 17.2**, we can write the polar form of a complex number as:

$$z = re^{i\theta}$$

5 Logarithmic Functions

The logarithm of a nonzero complex number is defined as the inverse of the exponential function, so $w = \ln z$.

Logarithm of a Complex Number

$$\ln z = \log_e |z| + i(\theta + 2k\pi)$$

6 Principal Value

In real calculus, $\log_e 5 = 1.6094$ has only one value, but in complex calculus, $\log_e 5 = 1.6094 + 2k\pi i$. The value of $\ln 5$ corresponding to $k = 0$ is called the **principal value** of $\ln 5$.

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2 \text{ and } \ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$$

CHAPTER 3

TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

1 Trigonometric Functions

Using Euler's formula, we see that the real functions of sine of cosine can be represented as a combination of exponential functions.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

2 Zeroes

To find zeroes of $\sin z$ and $\cos z$, we need to express both functions in the form $u + vi$. But first, recall that if y is real, then the hyperbolic sine and cosine functions are defined as:

$$\sinh y = \frac{e^y - e^{-y}}{2} \text{ and } \cosh y = \frac{e^y + e^{-y}}{2}$$

Fast forward to the definitions of $\sin z$ and $\cos z$:

$$\sin z = \sin x \cosh y + i \cos x + \sinh y$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

Part II

Differential Equations

CHAPTER 1

INITIAL VALUE PROBLEMS

1 Initial-Value Problems

Given some interval I containing the domain value x_0, the problem:

$$\text{Solve: } \frac{d^n y}{dx^n} = f(x, y, y', ..., y^{(n-1)})$$
$$\text{Subject to: } y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(n-1)}(x_0) = y_{n-1}$$

where all $y_0, ..., y_{n-1}$ values are specified constants, is called an initial-value problem. Each point $y(x_0) = y_0$ is called an initial condition.

2 First- and Second-Order IVPs

The problem described above is known as a n**th order initial-value problem**. First- and second-order IVPs are easy to interpret in geometric terms.

3 Existence and Uniqueness

In every initial-value problem we have two fundamental questions: Does a solution exist? If so, is it unique?

We have to be careful in using the words “a solution” versus “the solution“ because there may be multiple solutions, a single solution, or none at all.

CHAPTER 2

SOLUTION CURVES WITH NO SOLUTION

We start with two ways of analyzing a differential equation to get an approximate sense of that the solution curve looks like, without having to solve the equation.

1 Direction Fields

1.1 Slope

The slope of a tangent line at $(x, y(x))$ is the value of the first derivative evaluated at that point. If we take (x, y) to be any point in a region o the xy-plane where f is defined, then the value assigned to it by the function represents the slope of a line called the **lineal element**.

For example, consider the equation $y' = 0.2xy$, where $f(x, y) = 0.2xy$. At the point $(2, 3)$ the slope of a lineal element is $f(2, 3) = 0.2(2)(3) = 1.2$. Therefore, if a solution curve passes through the point, it does so tangent to the line segment – in other words, we can say the lineal element is a miniature tangent line.

1.2 Direction Field

If we draw a lineal element for f at each point (x, y) over a rectangular region, then we call this collection of lineal points a **direction or slope field**. The direction field suggests the shape of a family of curves, and it may be possible to notice certain points where the solutions act weird.

1.3 Increasing/Decreasing

If $y' > 0$ for all x in an interval I, then we know that the differentiable function $y = y(x)$ is increasing on I, and vice versa for $y' < 0$.

2 Autonomous First-Order DEs

2.1 DEs Free of the Independent Variable

An ordinary differential equation in which the independent variable does not appear explicitly is said to be autonomous. It is expressed normally as:

$$\frac{dy}{dx} = f(y)$$

2.2 Critical Points

The zeroes of the function $f(x)$ above are of special importance. We say a real number c is a critical point if it's a zero of f. If we plug in $y(x) = c$ into the above equation, then both sides equal zero.

If c is a critical point of $f(x)$, then $y(x) = c$ is a constant solution of the autonomous differential equation.

2.3 Solution Curves

Suppose a function has two critical points at c_1 and c_2, then the graphs of $y = c_1$ and $y = c_2$ are horizontal lines. These lines partition the region R into three subregions.
Here are some observations we can make without proof:

- If (x_0, y_0) is in any subregion R_1, R_2, R_3 and $y(x)$ is a solution whose graph passes through this point, then $y(x)$ stays in that subregion for all x. This is because the function may not cross equilibrium solutions.
- By continuity of f we have either $f(y) > 0$ or $f(y) < 0$ for all x. In other words, $f(y)$ cannot change signs in a subregion.
- Since $dy/dx = f(y(x))$ is either positive or negative in a subregion R_i, a solution $y(x)$ is strictly monotonic. This means that $y(x)$ can neither be oscillatory nor have local extrema.

2.4 Attractors and Repellers

Suppose $y(x)$ is a nonconstant solution of an autonomous DE and that c is a critical point of the DE; there are three types of behaviour $y(x)$ can exhibit near c.
In the diagram above, when both arrowheads on either side of c point towards c, then all solutions $y(x)$ that start from an initial point (x_0, y_0) near c will exhibit the behaviour $\lim_{x\to\infty} y(x) = c$. This critical point is said to be stable, or an attractor.

When both arrowheads face away from c, then all solutions will move away from c as x grows; this is known as an unstable critical point, or a repeller.

When both arrows point in the same direction, a solution $y(x)$ starting sufficiently near c will be attracted to c from one side and repelled from the other. These critical points are called semi-stable.

2.5 Translation Property

If $y(x)$ is a solution of an autonomous differential equation $dy/dx = f(y)$, then $y_1(x) = y(x-k)$, where k is a constant, is also a solution. Essentially, applying a horizontal translation to $y(x)$ will have no effect on the long-term behaviour of the function.

CHAPTER 3

LINEAR EQUATIONS

1 A Definition

A first-order DE of the form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be a linear equation in the variable y. If $g(x) = 0$, then the DE is also said to be homogeneous.

2 Standard Form

We can divide both sides of the above equation by $a_1(x)$ to obtain a more useful form of the DE:

$$\frac{dy}{dx} + P(x)y = f(x)$$

3 The Property

The DE given above has the property that its solution is the sum of the two solutions $y = y_c + y_p$, where y_c is a solution of the associated homogeneous equation:

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{d}{dx}[y_c + y_p] + P(x)[y_c + y_p] = \left[\frac{dy_c}{dx} + P(x)y_c\right] + \left[\frac{dy_p}{dx} + P(x)y_p\right] = f(x)$$

4 Method of the Integrating Factor

Check Lecture 2-2 notes to see this method of solving Linear First-Order ODEs.

1. Put a linear first-order equation into standard form then determine $P(x)$ and the integrating factor $e^{\int P(x)dx}$.

2. Multiply the standard form by the integrating factor. The left side is the derivative of the product of the integrating factor and y. Write:

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}f(x)$$

then integrate both sides of the equation.

4.1 Singular Points

When we divide by $a_1(x)$ to get the standard form, we need to be careful about what happens for values of x where $a_1(x) = 0$. The discontinuity may carry over to functions in the general solution of the differential equation.

5 Error Function

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt + \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt = 1$$

where the first part is $erf(x)$ and the second part is $erfc(x)$. We also see that:

$$erf(x) + erfc(x) = 1$$

CHAPTER 4

THEORY OF LINEAR EQUATIONS

1 Initial-Value and Boundary-Value Problems

1.1 Boundary-Value Problem

Another type of problem which entails solving a linear differential equation or order 2 or more, where the dependent variable y or its derivatives are specified at different points.

$$\begin{aligned} \textbf{Solve:} \quad & a_2(x)\frac{d^y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \\ \textbf{Subject to:} \quad & y(a) = y_0, y(b) = y_1 \end{aligned}$$

This sort of problem is called a boundary-value problem because the prescribed values $y(a) = y_0$ and $y(b) = y_1$ are called boundary conditions.

We may define general boundary conditions as:

$$A_1 y(a) + B_1 y'(a) = C_1$$
$$A_2 y(a) + B_2 y'(b) = C_2$$

2 Homogeneous Equations

$$a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

The above DE is homogeneous.

$$a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

The above DE is non-homogeneous. In order to solve the above non-homogeneous DE, we will first need to solve the associated heterogeneous equation.

2.1 Differential Operators

Differentiation is denoted by the capital letter D, such that $dy/dx = Dy$. Higher-order derivatives can be expressed in a similar manner. Remember from Calculus II that differentiation matches the criteria for linear transformations.

2.2 Superposition Principle

Theorem 3.1.2 – Superposition Principle for Homogeneous Equations

The sum, or **superposition** of two or more solutions to a homogeneous DE is also a solution.

2.3 Linear Dependence and Linear Independence

We carry over the same definitions of linear dependence and Independence from linear algebra.

2.4 Solutions of Differential Equations

Suppose each of the functions $f_1(x), f_2(x), ..., f_n(x)$ possesses at least $n-1$ derivatives. The determinant of the matrix

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$$

is called the Wronskian of the function.

Theorem 3.1.3 – Criterion for Linearly Independent Solutions

Let $y_1, y_2, .., y_n$ be n solutions of the homogeneous linear nth-order DE on an interval I. Then the set of solutions is linearly independent on I iff $W(y_1, y_2, ..., y_n) \neq 0$ for every x in the interval.

Definition 3.1.3 – Fundamental Set of Solutions

Any set of $y_1, y_2, ..., y_n$ of n linearly independent solutions of the homogeneous linear nth-order DE on an interval I is said to be a fundamental set of solutions on that interval.

Theorem 3.1.4 – Existence of a Fundamental Set

There exists a fundamental set of solutions for the homogeneous linear nth-order DE on an interval I.

Theorem 3.1.5 – General Solutions for Homogeneous Equations

The general solution of the equation on the interval I is given by:

$$y = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x)$$

3 Nonhomogeneous Equations

General Solution for Nonhomogeneous Equations

The general solution of the equation on the interval I is:

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p(x)$$

The linear combination part of the solution is called the complementary function. This gives us:

$$y = y_c + y_p$$

CHAPTER 5

REDUCTION OF ORDER

In Chapter 3.1 we saw that the general solution of a homogeneous linear second-order differential equation:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{5.1}$$

was a linear combination $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I. We will learn a method to find solutions in the next section. It turns out we can construct a second solution y_2 od a homogeneous equation (1) provided we know one nontrivial solution y_1 of the DE.

1 Reduction of Order

Suppose $y(x)$ denotes a known solution of equation (1). We seek a second solution $y_2(x)$ of (1) that is linearly independent to $y_1(x)$. The idea is to find $u(x)$ where $y_2(x) = u(x)y_1(x)$, this method is called reduction of order, since we can reduce a second-order ODE into a first-order ODE.

Let us explore this using an example:

Example: Finding a Second Solution

Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$ use reduction of order to find a second solution y_2.

If $y = u(x)y_1(x) = u(x)e^x$, then the first two derivatives of y are obtained from the product rule:

$$y' = ue^x + e^xu', y'' = ue^x + 2e^xu' + e^xu''$$

By substituting y and y'' into the original DE, we can simplify to:

$$y'' - y = e^x(u'' + 2u') = 0$$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$. If we make the substitution $w = u'$, this linear second-order equation in u becomes $w' + 2w = 0$, which is a linear first-order equation in w. Using the integrating factor e^{2x}, we can write $d/dx[e^{2x}w] = 0$. After integrating we have $w = c_1e^{-2x}$, which becomes $u' = c_1e^{-2x}$. Integrating again gives us: $u = -\frac{1}{2}c_1e^{-2x} + c_2$. Thus, we have:

$$y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2e^x \tag{5.2}$$

By choosing $c_2 = 0$ and $c_1 = -2$, we obtain the desired $y_2 = e^{-x}$.

2 General Case

Suppose we divide by $a_2(x)$ in order to put equation (1) in the standard form:

$$y'' + P9x)y' + Q(x)y = 0 \tag{5.3}$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Suppose further that $y_1(x)$ is a solution of (3) on I and that $y_1(x) \neq 0$ for every x in the interval. If we define $y = u(x)y_1(x)$ it follows that:

$$\begin{aligned} &y' = uy_1' + y_1u', \\ &y'' = uy_1'' + 2y_1'u' + y_1u'' \\ &y'' + Py' + Qy = u[y_1'' + Py_1' + Qy_1] + y_1u'' + (2y_1' + Py_1)u' = 0 \end{aligned}$$

Implying that we have:

$$y_1u'' + (2y_1' + Py_1)u' = 0 \text{ or } y_1w' + (2y_1' + Py_1)w = 0 \tag{5.4}$$

if have $w = u'$. Observe that the last equation in (4) is both linear and separable. We can then separate variables and integrate:

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + Pdx = 0$$

$$\ln|wy_1^2| = -\int Pdx + c \text{ or } wy_1^2 = c_1e^{-\int Pdx}$$

We can solve the last equation for w using $w = u'$ and integrating again:

$$u = c_1\int \frac{e^{-\int Pdx}}{y_1^2}dx + c_2$$

Theorem 3.2.1 Reduction of Order

Let $y_1(x)$ be a solution of the homogeneous differential equation (3) on an interval I and that $y_1(x) \neq 0$ for all x in I. Then:

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2} \tag{5.5}$$

is a second solution of (3) on the interval I.

To prove linear independence of the functions, we can use **Theorem 3.1.3 The Wronskian** of the function is:

$$W(y_1, y_2) = det\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = e^{-\int P(x)dx \neq 0}$$

CHAPTER 6

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

We will explore whether exponential solutions exist for homogeneous linear higher-order ODEs such as:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0 \tag{6.1}$$

Where all coefficients a_i are real and $a_n \neq 0$.

1 Auxiliary Equation

We can construct an auxiliary equation, which takes us from:

$$ay'' + by' + cy = 0 \tag{6.2}$$

to this:

$$am^2 + bm = c = 0 \tag{6.3}$$

1.1 Case I: Distinct Real Roots

If we have two unequal real roots m_1 and m_2, then we have two solutions: $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$. We see that these functions must be linearly independent over x, and form a fundamental set.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \tag{6.4}$$

1.2 Case II: Repeated Real Roots

When $m_1 = m_2$ we obtain only one exponential solution. The general solution is:

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} \tag{6.5}$$

1.3 Case III: Complex Conjugate Roots

If we have a negative determinant then the roots m_1 and m_2 are complex. We use Euler's formula, and with a little manipulation we end up with:

$$y = e^{ax}(c_1 \cos \beta x + c_2 \sin \beta x \tag{6.6}$$

2 Two Equations Worth Knowing

The two linear DEs

$$y'' + k^2 y = 0 \text{ and } y'' - k^2 y = 0$$

are important in applied mathematics. $y'' + k^2 y = 0$ has imaginary roots $m_1 = ki$ and $m_2 = -ki$ for its auxiliary equation, and the general solution of the DE is:

$$y = c_1 \cos kx + c_2 \sin kx \tag{6.7}$$

Likewise, the auxiliary equation for $y'' - k^2 y = 0$ has distinct real roots $m_1 = k$ and $m_2 = -k$, and the general solution of the DE is:

$$y = c_1 e^{kx} + c_2 e^{-kx} \tag{6.8}$$

Notice if we choose $c_1 = \frac{1}{2}$ and $c = \frac{1}{2}$ respectively, we end up with $y = \frac{1}{2} \cosh kx$ and $y = \frac{1}{2} \sinh kx$. An alternative form of this general solution would be:

$$y = c_1 \cosh kx + c_2 \sinh kx \tag{6.9}$$

3 Higher-Order Derivatives

Generally, to solve an n-th order DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0 \tag{6.10}$$

we must solve the n-th degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + ... + a_1 m + a_0 = 0 \tag{6.11}$$

CHAPTER 7

VARIATION OF PARAMETERS

This method is named after the Italian astronomer and mathematician Joseph-Louis Lagrange.

1 Some Assumptions

We start with the linear second-order DE:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \tag{7.1}$$

and put it in standard form:

$$y'' + P(x)y' + Q(x)y = f(x) \tag{7.2}$$

2 Method of Variation of Parameters

We seek a solution of the form:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{7.3}$$

where y_1 and y_2 form a fundamental set of solutions on I. We then differentiate y_p twice to get:

$$\begin{aligned} y_p' &= u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2' \\ y_p'' &= u_1y_1'' + y_1'u_1' + y_1u_1'' + u_1'y_1' + u_2y_2'' + y_2'u_2' + y_2u_2'' + u_2'y_2' \end{aligned}$$

Substituting (3) and its derivatives into (2) and grouping terms together yields:

$$y_p'' + P(x)y_p' + Q(x)y_p = \frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x) \tag{7.4}$$

By Cramer's Rue, we can solve the system:

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$

can be expressed in terms of determinants

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \text{ and } u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W} \tag{7.5}$$

where:

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}, W_1 = \det \begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix}, W_2 = \det \begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix} \tag{7.6}$$

3 Constants of Integration

When computing the infinite integrals of u_1' and u_2', we don't need to introduce any constants. This is because:

$$\begin{aligned} y = y_c + y_p &= c_1 y_1 + c_2 y_2 + (u_1 + a_1) y_1 + (u_2 + b_1) y_2 \\ &= (c_1 + a_1) y_1 + (c_2 + b_1) y_2 + u_1 y_1 + u_2 y_2 \\ &= C_1 y_1 + C_2 y_2 + u_1 y_1 + u_2 y_2 \end{aligned}$$

4 Integral-Defined Functions

Going back to (3), we can use the following to solve linear second-order DE:

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(t) f(t)}{W(t)} dt \text{ and } u_2(x) = \int_{x_0}^{x} \frac{y_1(t) f(t)}{W(t)} dt$$

5 Higher-Order Equations

Non-homogeneous second order equations can be put into the form:

$$y^{(n)} + P_{n-1}(x) y^{(n-1)} + ... + P_1(x) y' + P_0(x) y = f(x) \tag{7.7}$$

If $y_c = c_1 y_1 + ... + c_n y_n$ is the complementary function, then a particular solution is:

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) + ... + u_n(x) y_n(x)$$

Cramer's Rule gives us:

$$u_k' = \frac{W_k}{W}, k = 1, 2, ...n$$

Part III

The Laplace Transform

CHAPTER 1

DEFINITION OF THE LAPLACE TRANSFORM

1 Integral Transform

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b\to\infty}\int_0^b K(s,t)f(t)dt \tag{1.1}$$

If the limit in (1) exists, we say the integral exists, or is convergent; else, it does not exist and is divergent.

2 A Definition

The function $K(s,t)$ is said to be the kernel of the transform. The choice of $K(s,t) = e^{-st}$ as the kernel gives an important integral transform.

Laplace Transform

$$F(s) = \int_0^\infty e^{-st}f(t)dt \tag{1.2}$$

is said to be the Laplace transform of f. The domain of $F(s)$ is the set of values for s for which the improper integral (2) converges.

We may denote the Laplace transform in many ways:

$$\mathcal{L}\{f(t)\} = F(s), \mathcal{L}\{g(t)\} = G(s), \mathcal{L}\{i(t)\} = I(s)$$

3 Linearity of the Laplace Transform

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} \tag{1.3}$$

4 Sufficient Conditions for Existence of $\mathcal{L}\{f(t)\}$

Definition – Exponential Order

A function f is said to be of exponential order is there are constants $c, M > 0$ and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Theorem 4.1.2 – Sufficient Conditions for Existence

If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

CHAPTER 2

INVERSE TRANSFORMS AND TRANSFORMS OF DERIVATIVES

1 Inverse Transforms

1.1 Linearity of the Inverse Laplace Transformation

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\} \tag{2.1}$$

1.2 Transform of a Derivative

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^{t_0} e^{-st} f'(t)dt + \int_{t_0}^{\infty} e^{-st} f'(t)dt \\
&= \left[e^{-st} f(t) \Big|_0^{t_0} + s\int_0^{t_0} e^{-st} f(t)dt \right] + \left[e^{-st} f(t) \Big|_{t_0}^{\infty} + s\int_{t_0}^{\infty} e^{-st} f(t)dt \right] \\
&= -f(0) + \lim_{t\to\infty} e^{-st} f(t) + s\left[\int_0^{t_0} e^{-st} f(t)dt + \int_{t_0}^{\infty} e^{-st} f(t)dt \right] \\
&= -f(0) + \lim_{t\to\infty} e^{-st} f(t) + s\int_0^{\infty} e^{-st} f(t)dt
\end{aligned}$$

We may condense this into:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \tag{2.2}$$
$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \tag{2.3}$$
$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0) \tag{2.4}$$

2 Solving Linear ODEs

$$a_n\mathcal{L}\{\frac{d^ny}{dt^n}\} + a_{n-1}\mathcal{L}\{\frac{d^{n-1}y}{dt^{n-1}}\} + ... + a_0\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \tag{2.5}$$

From (5) we get:

$$a_n[s^nY(s) - s^{n-1}y(0) - ... - y^{(n-1)}(0)] + ... + a_0Y(s) = G(s) \tag{2.6}$$

Essentially this states that the Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in $Y(s)$. If we solve the transformed function (6) for $Y(s)$, we obtain $P(s)Y(s) = Q(s) + G(s)$, then:

$$Y(s) = \frac{Q(s) + G(s)}{P(s)} \tag{2.7}$$

Theorem 4.2.3 – Behaviour of $F(s)$ as $s \to \infty$

If a function f is piecewise continuous on $[0, \infty)$ and of exponential order with c as specified in Definition 4.1.2 and $\mathcal{L}\{f(t)\} = F(s)$, then $\lim_{s\to\infty} F(s) = 0$

CHAPTER 3

TRANSLATION THEOREMS

1 Translation on the s-axis

In general, if we know $\mathcal{L}\{f(t)\} = F(s)$, it is possible to compute the Laplace transform of an exponential multiple of the function f with no additional effort other than translating or shifting $F(s)$ to $F(s-a)$.

Theorem 4.3.1 – First Translation Theorem

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Inverse Form of Theorem 4.3.1

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

2 Translation on the t-axis

2.1 Unit Step Function

It is convenient to define a special function that is 0 until a specified time a, then 1 after that time. This is also called the **Heaviside function**.

Definition 4.3.1 – Unit Step Function

$$\mathcal{U}(t-a) = \begin{cases} 0, 0 \leq t \leq a \\ 1, t \geq a \end{cases}$$

This definition implies that the function

$$f(t) = \begin{cases} g(t), 0 \leq t < a \\ h(t), t \geq a \end{cases}$$

is the same as the function

$$f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$$

Theorem 4.3.2 Second Translation Theorem

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

Inverse Form of Theorem 4.3.2

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

Using Definition 4.1.1, we can derive an alternative version of the above theorem.

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

CHAPTER 4

ADDITIONAL PROPERTIES

1 Derivatives of Transforms

1.1 Multiplying a Function by t^n

Theorem 4.4.1 – Derivatives of Transforms

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

2 Transforms of Integrals

2.1 Convolution

$$f * g = \int_0^t f(\tau) g(t-\tau) d\tau$$

Theorem 4.4.2 – Convolution Theorem

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

Transform of an Integral

$$\mathcal{L}\{\int_0^t f(\tau) d\tau\} = \frac{F(s)}{s} \tag{4.1}$$

The convolution theorem is useful in solving other types of equations where an unknown function appears under an integral sign. For example we have the Volterra integral equation

$$f(t) = g(t) + \int_0^t f(\tau) h(t-\tau) d\tau$$

3 Transform of a Periodic Function

Theorem 4.4.3 – Transform of a Periodic Function

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T f(t)dt$$

CHAPTER 5

THE DIRAC DELTA FUNCTION

1 Unit Impulse

If we graph the piecewise function

$$\delta_a(t-t_0) = \begin{cases} 0, 0, \leq t < t_0 - a \\ \frac{1}{2a}, t_0 - 1 \leq t < t_0 + a \\ t \geq t_0 + a \end{cases} \tag{5.1}$$

we get the following: The function is called a unit impulse since we have defined it such that:

$$\int_0^\infty (t-t_0)dt = 1$$

2 The Dirac Delta Function

Theorem 4.5.1 – Transform of the Dirac Delta Function

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

We can conclude from the above theorem that:

$$\mathcal{L}\{\delta(t)\} = 1$$

2.1 Alternative Definitions

If f is a continuous function, then

$$\int_0^\infty f(t)\delta(t-t_0)dt = f(t_0)$$

This result of $\delta(t-t_0)$ is known as the sifting property.

Part IV

Complex Analysis

CHAPTER 1

SETS IN THE COMPLEX PLANE

1 Terminology

Suppose $z_0 = x_0 + iy_0$. Since $|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ is the distance between points $z = x + iy$ and $z_0 = x_0 + iy_0$ that satisfy $|z - z_0| = \rho$ for $\rho > 0$ lie on a disk of radius ρ centered at z_0.

The points that satisfy $|z - z_0| < \rho$ are within the disk but not on it. This set is called the neighbourhood of z_0 or an open disk. An **interior point** of a set S is a point for which there exists some neighbourhood of z_0 that lies entirely within S. If every point in S is an interior point, then S is an **open set**.

If every neighbourhood of z_0 has at least one point of S and one point not in S, then it is a **boundary point**.

If any pair of points z_1 and z_2 in an open set S can be connected by a polygon line that lies entirely within the set, then the set is called **connected**. A **region** is a domain in the complex plane with all, some, or none of its boundary points. A region containing all of its boundary bounds is labelled **closed**.

CHAPTER 2

FUNCTIONS OF A COMPLEX VARIABLE

1 Functions of a Complex Variable

A function of a complex variable of simply complex function is defined as:

$$w = f(z) = u(x, y) + iv(x, y)$$

2 Limits and Continuity

Definition 17.4.1 – Limit of a Function

Suppose the function f is defined in some neighbourhood of z_0 except possibly at z_0 itself. Then, f is said to possess a limit, written as:

$$\lim_{z \to z_0} f(z) = L$$

Definition 17.4.2 – Continuity at a Point

A function f is continuous at a point z_0 if:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

3 Derivatives

The derivative of a complex function is defined as:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

If the above limit exists, then the function is said to be differentiable at z_0. As in real variables, differentiability implies continuity: "If f is differentiable at z_0, then f is continuous at z_0."

CHAPTER 3

CONTOUR INTEGRALS

1 A Definition

A piecewise smooth curve C is called a contour or path. An integral of $f(z)$ on C is denoted by $\int_C f(z)$ or $\oint_C f(z)$ if the contour is closed.

Definition 18.1.1 – Contour Integral

The contour integral of f along C is:

$$\int_C f(z)dz = \lim_{||P||\to 0} \sum_{k=1}^{n} f(z)_k^* \Delta z_k \tag{3.1}$$

2 Method of Evaluation

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udx \tag{3.2}$$

In other words, we see that the contour integrals is a combination of two real line integrals. Thus, we arrive at the following theorem:

Theorem 18.1.1 – Evaluation of a Contour Integral

If f is continuous on a smooth curve C given by $z = x + it$, then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt \tag{3.3}$$

3 Circulation and Net Flux

When we interpret the complex function $f(z) = u(x, y) + iv(x, y)$ as a vector, we get the following:

$$\oint_C f \cdot \mathbf{T}ds = \oint_C udx + vdy \tag{3.4}$$

$$\oint_C f \cdot \mathbf{n} ds = \oint_C u dy - i dx \tag{3.5}$$

The integral in (4) is called the **circulation** around C, which measures the tendency of the flow to rotate the curve C. The integral in (5) is called the **net flux** across C, which measures the presence of sources or sinks for the fluid inside C. We see that:

$$\left(\oint_C f \cdot \mathbf{T} ds\right) + i\left(\oint_C f \cdot \mathbf{n} ds\right) = \oint_C (u - iv)(dx + idy) = \oint f(\bar{z}) dz \tag{3.6}$$

CHAPTER 4

INDEPENDENCE OF PATH

In real calculus we have the Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{4.1}$$

In this case we can say that the line integral is independent of the path. So now we ask the question: is there a complex version of the Fundamental Theorem of Calculus.

1 Path Independence

Definition 18.3.1 – Independence of Path

Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ is said to be independent of the path if its value is the same for all contours C in D. We see from the Cauchy-Goursat Theorem that

$$\int_C f(z)dz + \int_{-C_1} f(z)dz = 0 \tag{4.2}$$

which is equivalent to

$$\int_C f(z)dz = \int_{C_1} f(z)dz \tag{4.3}$$

Theorem 18.3.1 – Analyticity Implies Path Independence

If f is an analytic function in a simply connected domain D, then $\int_C f(z)dz$ is independent of the chosen path C.

Definition 18.3.2 – Antiderivative

Suppose f is continuous in a domain D. If there exists a function F such that $F'(z) = f(z)$ for each z in D, then F is called an **antiderivative** of F.

Theorem 18.3.2 – Fundamental Theorem for Contour Integrals

Suppose f is continuous in a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point z_0 and terminal point z_1,

$$\int_C f(z)dz = F(z_1) - F(z_0) \tag{4.4}$$

In proving the above theorem, we can make the following two statements:

1. If a continuous function f has an antiderivative F in D, then $\int_C f(z)dz$ is independent of the path.

2. If f is continuous and $\int_C f(z)dz$ is independent of the path in a domain D, then f has an antiderivative everywhere in D.

Theorem 18.3.3 – Existence of an Antiderivative

If f is analytic in a simply connected domain D, then f has an antiderivative in D; that is, there exists a function F such that $F'(z) = f(z)$ for all z in D.

CHAPTER 5

CAUCHY INTEGRAL FORMULA

The most significant result of the Cauchy-Goursat Theorem is that the value of an analytic function f at any point z_0 in a simply connected domain can be represented by a contour integral. We will further show that an analytic function f in a simply connected domain possesses derivatives of all orders.

1 First Formula

Theorem 18.4.1 – Cauchy's Integral Formula

Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \tag{5.1}$$

2 Second Formula

Theorem 18.4.2 – Cauchy's Integral Formula for Derivatives

Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{5.2}$$

3 Liouville's Theorem

If we take the contour $C$to be the circle $|z - z + 0| = r$, it follows from (2) that:

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n} \tag{5.3}$$

This result is called Cauchy's inequality, and we can use it to prove the following theorem.

Theorem 18.4.3 – Liouville's Theorem

The only bounded entire functions are constants.

4 Fundamental Theorem of Algebra

If $P(z)$ is a nonconstant polynomial, then the equation $P(z) = 0$ has at least one root.

CHAPTER 6

SEQUENCES AND SERIES

1 Sequences

A sequence is a function whose domain is the set of positive integers. If $\lim_{n\to\infty} z_n = L$, we say the sequence is convergent.

Theorem 19.1.1 – Criterion for Convergence

A sequence $\{z_n\}$ converges to a complex number L iff $Re(z_n)$ converges to $Re(L)$ and $Im(z_n)$ converges to $Im(L)$.

2 Series

An infinite series of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + ... + z_n + ...$$

is convergent if the sequence of partial sums $\{S_n\}$ where

$$S_n = z_1 + z_2 + ... + z_n$$

converges. If $S_n \to L$ as $n \to \infty$, we say the sum of the series is L.

3 Geometric Series

For the geometric series

$$\sum_{k=1}^{\infty} ak^{k-1} = a + az + az^2 + ... + az^{n-1} + ...$$

the nth term of the sequence of partial sums is

$$S_n = a + az + az^2 + ... + az^{n-1}$$

Solving for S_n gives

$$S_n = \frac{a(1-z^n)}{1-z} \tag{6.1}$$

Since $z^n \to 0$ as $n \to \infty$, whenever $|z| < 1$, we conclude that the series converges to

$$\frac{a}{1-z} \tag{6.2}$$

Theorem 19.1.2 – Necessary Conditions for Convergence

If $\sum_{k=1}^{\infty} z_k$ converges, then $lim_{n\to\infty} z_n = 0$.

Theorem 19.1.3 – The n-th Term Test for Divergence

If $\lim_{n\to\infty} z_n \neq 0$, then the series diverges.

Theorem 19.1.1 – Absolute Convergence

An infinite series is said to be absolutely convergent if the equivalent series with the sum term with absolute value brackets also converges.

Theorem 19.1.4 – Ratio Test

Suppose we have an infinite series such that

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L \tag{6.3}$$

Then:

1. If $L < 1$ the series converges absolutely.
2. If $L > 1$ or $L = \infty$, then the series diverges.
3. If $L = 1$ the test is inconclusive

Theorem 19.1.5 – Root Test

Suppose we have an infinite series such that

$$\lim_{n\to\infty} = \sqrt[n]{|z_n|} = L \tag{6.4}$$

Then:

1. If $L < 1$ the series converges absolutely.
2. If $L > 1$ or $L = \infty$, then the series diverges.
3. If $L = 1$ the test is inconclusive

4 Power Series

An infinite series of the form

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ... \tag{6.5}$$

where coefficients a_k are complex coefficients is called a power series. The power series is said to be centered at z_0.

5 Circle of Convergence

Every complex power series has a radius of convergence R. It also has a circle of convergence defined by $|z-z_0| = R$. The power series converges absolutely when $|z-z_0| < R$ and diverges for $|z-z_0| > R$. The radius R can be:

1. 0,
2. a finite number, or
3. infinity.

CHAPTER 7

TAYLOR SERIES

Theorem 19.2.1 – Continuity

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ represents a continuous function f within it's circle of convergence $|z - z_0| = R$.

Theorem 19.2.2 – Term-by-Term Integration

A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be integrated term by term within its circle of convergence $|z - z_0| = R$ for every contour C lying entirely within the circle of convergence.

Theorem 19.2.3 – Term-by-Term Differentiation

A power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ can be differentiated term by term within it's circle of convergence $|z - z_0| = R$.

1 Taylor Series

A power series can represent an analytic function within it's circle of convergence. The series:

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + ... = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \tag{7.1}$$

is called the Taylor series for f centered at z_0. When the Taylor series is taken at $z_0 = 0$, then we may call it a Mclaurin series (though we don't really do that).

Theorem 19.2.4 – Taylor's Theorem

Let f be analytic within a domain D and let z_0 be a point D. Then f has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \tag{7.2}$$

valid for the largest circle C with centre at z_0 and radius that lies entirely within D.

CHAPTER 8

LAURENT SERIES

If a complex function f is not analytic at a point $z = z_0$, then we call that point a **singularity** or **singular point** of the function.

1 Isolated Singularities

Suppose $z = z_0$ is a singularity of a complex function f. The point $z = z_0$ is said to be an **isolated singularity** of the function f if there exists some deleted neighbourhood or punctured disk of z_0 throughout which f is analytic.

2 A New Kind of Series

Let us start with

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{8.1}$$

where the left-hand sum is called the **principal part** of the series and the right-hand sum is called the **analytic part** of the series.
The sum of the two parts converges when z is in an annular domain defined by $r < |z - z_0| < R$.

Theorem 19.3.1 – Laurent's Theorem

Let f be analytic within the annular domain D defined by $r < |z - z_0| < R$. Then, f has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{8.2}$$

valid for $r < |z - z_0| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds \tag{8.3}$$

CHAPTER 9

ZEROES AND POLES

1 Classification of Isolated Singular Points

An isolated singular point $z = z_0$ of a complex function f is given a classification depending on whether the principal part of it's Laurent expansion contains zero, a finite number, or an infinite number of terms.

1. If the principal part is zero, or, all coefficients a_k are zero, then z_0 is called a **removable singularity**.
2. If the principal part contains a finite number of nonzero terms, then z_0 is called a **pole**. If the last nonzero coefficient is a_n, $n \geq 1$, then we say that z_0 is a pole of order n. If z_0 is a pole of order 1, then the pol is called a **simple pole**.
3. If the principal part contains infinitely many nonzero terms, then z_0 is called an **essential singularity**.

2 Zeros

Theorem 19.4.1 – Pole of Order n

If the functions f and g are analytic at z_0 and f has a zero of order n at z_0 and $g(z_0) \neq 0$ then the function $F(z) = g(z)/f(z)$ has a pole of order n at z_0.

CHAPTER 10

THE CAUCHY RESIDUE THEOREM

1 Residue

The coefficient a_{-1} of $1/(z - z_0)$ in the Laurent series given above is called the **residue** of the function f at the isolated singularity z_0. We shall use the notation

$$a_{-1} = \text{Res}(f(z), z_0) \tag{10.1}$$

Theorem 19.5.1 – Residue at a Simple Pole

If f has a simple pole at z_0, then

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{10.2}$$

Theorem 19.5.2 – Residue at a Pole of Order n

If f has a pole of order n at z_0, then

$$\text{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \tag{10.3}$$

Here is an alternative method for computing a residue at a *simple pole*.

$$\text{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)} \tag{10.4}$$

2 Residue Theorem

Theorem 19.5.3 – Cauchy's Residue Theorem

Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of singular points z_1, z_2, ..., z_n within C, then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k) \tag{10.5}$$

CHAPTER 11

EVALUATION OF REAL INTEGRALS

In this section we will see how residue theory can help us real integrals of the forms:

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta \tag{11.1}$$

$$\int_{-\infty}^{\infty} f(x)dx \tag{11.2}$$

$$\int_{-\infty}^{\infty} f(x)\cos\alpha x dx \text{ and } \int_{-\infty}^{\infty} f(x)\sin\alpha x dx \tag{11.3}$$

1 Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

Convert an integral form into a complex integral where the contour C is the unit circle centered at the origin.

$$dz = ie^{i\theta}d\theta, \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \cos\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \tag{11.4}$$

we replace, $d\theta, \cos\theta, \sin\theta$ with

$$d\theta = \frac{dz}{iz}, \cos\theta = \frac{1}{2}(z+z^{-1}), \sin\theta = \frac{1}{2i}(z-z^{-1}) \tag{11.5}$$

The integral in question then becomes

$$\oint_C F\left(\frac{1}{2}(z+z^{-1}), \frac{1}{2i}(z-z^{-1})\right)\frac{dz}{iz} \tag{11.6}$$

2 Integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$

When f is continuous throughout the domain, recall from calculus that the improper integral is defined in terms of two limits.

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{0} f(x)dx + \lim_{R\to\infty}\int_0^R f(x)dx \tag{11.7}$$

If both limits exist, the integral is said to be convergent, otherwise, the integral is divergent. If the limit is convergent, we can use a single integral:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \tag{11.8}$$

The limit is called the Cauchy Principal Value of the integral and is written as

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \tag{11.9}$$

To evaluate an integral $\int_{-\infty}^{\infty} f(x)dx$, where $f(x) = P(x)/Q(x)$ is continuous on $(-\infty, \infty)$, by residue theory we replace x by the complex value z and integrate the complex function f over a closed contour C, that consists of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(x) = P(z)/Q(z)$ in the upper half-plane $\Re(z) > 0$.

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k) \tag{11.10}$$

Theorem 19.6.1 – Behaviour of Integral as $R \to \infty$

Suppose $f(z) = P(z)/Q(z)$ where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \geq n+2$. If C_R is a semicircular contour $z = \Re e^{ie}$, $0 \leq \theta \leq \pi$ then $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

3 Integrals of the Forms $\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$ and $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$

When $\alpha > 0$, these integrals are referred to as **Fourier integrals**. Using Euler's formula $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$, we get:

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x dx \tag{11.11}$$

Before continuing, we give without proof sufficient conditions under which the contour integral along C_R approaches zero as $R \to \infty$.

4 Indented Contours

The improper integrals that we have considered are continuous throughout their domain. If f has poles on the real axis, we use an indented contour. The symbol C_r denotes a semicircle contour centered at $z = c$ oriented in the positive direction. The next theorem is important to this discussion.

Theorem 19.6.3 – Behaviour of Integral as $r \to 0$

Suppose f has a simple pole $z = c$ on the real axis. If C_r is the contour defined $z = c + re^{i\theta}$, then

$$\lim_{r\to 0} \int_{C_r} f(z)dz = \pi i \ \text{Res}(f(z), c) \tag{11.12}$$