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Introduction

1 Signals and Systems

1.0 Introduction

- Begin our development of analysis for signals and systems
    - Introducing mathematical descriptions and representations

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2 Linear Time-Invariant Systems

2.0 Introduction

- Two properties: linearity and time-invariance
- Many physical processes posses these properties and can be modelled as **linear time-invariant (LTI) systems**
- LTI systems also posses the property of superposition
- We can characterize any LTI system's response to a unit impulse
    - Convolution sum for discrete-time signals and convolution integral for continuous-time signals
- We then consider class of continuous- and discrete-time signals described by linear constant-coefficient DEs
- Lastly we will examine the continuous-time unit impulse function and other functions

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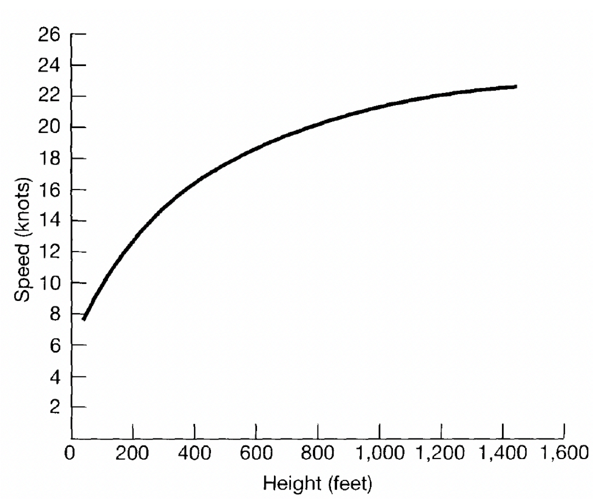
Fundamentals of Continuous- and Discrete-Time Signals

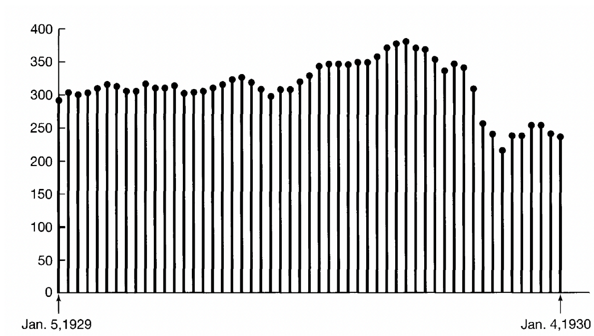
1 Signals and Systems

1.1 Continuous-Time and Discrete-Time Signals

1.1.1 Examples and Mathematical Representation

- Signals can be represented in many ways, but the information in a signal is contained as a pattern of variations.
- Signals are represented mathematically as equations in one or more variables
- We will consider two basic types of signals
    - One where the independent variable is continuous — continuous-time signals, and
    - One where the independent variable is discrete — discrete-time signals





- Conventionally, we use the variable t to represent continuous independent variables, and n to represent discrete variables
    - $x(t)$
    - $x[n]$
- To emphasize the fact that discrete-time signals are only defined for integer values, we sometimes call them discrete-time *sequences*

- One can derive a discrete-time signal by sampling a continuous-time signal at regular intervals

1.1.2 Signal Energy and Power

If $v(t)$ and $i(t)$ are the voltage and current across a resistor, then the instantaneous power is given by:

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

The total energy dissipated by over the interval $t_1 < t < t_2$ is

$$\frac{1}{R}\int_{t_1}^{t_2} v^2(t)\;dt$$

and the average power over this interval is

$$\frac{1}{t_2 - t_1}\frac{1}{R}\int_{t_1}^{t_2} v^2(t)\;dt$$

We can define the time-averaged power over an infinite interval as

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\;dt = \lim_{T\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{+N}|x[n]|^2$$

1.2 Transformations of the Independent Variable

1.2.1 Examples of Transformations of the Independent Variable

- In this section we will focus on a number of elementary signal transformations that modify the independent variable (time axis)
- This will allow us to discuss important basic properties of signals and systems
- The simplest transformation is the **time shift**, represented mathematically as $x(t - t_0)$ or $x[n - n_0]$. It represents a translation along the independent axis
- Another transformation is **time reversal**, which is represented by $x(-t)$ or $x[-n]$ and can be obtained by reflecting the signal along the dependent axis
- Then there is **time scaling**, which is represented by $x[\alpha n]$ or $x(\alpha t)$.

1.2.2 Periodic Signals

- A signal is **periodic** if there is a positive value T such that $x(t) = x(t + T)$ for all t.
  - We call x periodic with period T
  - The smallest T_0 for which the above identity holds is called the **fundamental period** T_0.

1.2.3 Even and Odd Signals

A signal is called **odd** if

$$x(-t) = x(t)$$
$$x[-n] = x[n]$$

and **even** if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

- Every signal can be decomposed into even and odd component signals

1.3 Exponential and Sinusoidal Signals

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

The continuous-time complex exponential signal is of the form

$$x(t) = Ce^{at}$$

If C and a are real, then the signal is a real exponential. If a is purely imaginary, then consider

$$x(t) = e^{j\omega_0 t}$$

For $x(t)$ to be periodic, we must have a period T such that

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t}e^{j\omega_0 T}$$

where $e^{j\omega_0 T} = 1$. Then, the smallest possible value for which $x(t)$ is still periodic is

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Furthermore, by using Euler's formula, we can write

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

The inverse of the fundamental period T_0 is called the **fundamental frequency** ω_0.

1.3.3 Periodicity Properties of Discrete-Time Complex Exponentials

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m
Fundamental frequency ω_0	Fundamental frequency ω_0/m
Fundamental period $\omega_0 = 0: \text{ undefined}$	Fundamental period $\omega_0 = 0: \text{ undefined}$

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
$\omega_0 \neq 0 : \frac{2\pi}{\omega_0}$	$\omega_0 \neq 0 : \frac{2\pi m}{\omega_0}$

1.4 The Unit Impulse and Unit Step Functions

1.4.1 The Discrete-Time Unit Impulse and Unit Step Functions

The **unit impulse** or **unit sample** is defined as

$$\delta[m] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$

The **unit step** function, another basic discrete-time signal, is defined as

$$u[n] = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}$$

We can see that the discrete-time unit impulse is the first difference of the discrete-time unit step functions

$$\delta[n] = u[n] - u[n-1]$$

Conversely, the discrete-time unit step is the running sum of the unit sample

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

1.4.2 The Continuous-Time Unit Step and Unit Impulse Functions

The unit step in continuous-time is defined similarly to its discrete-time counterpart. One difference is that the unit step function in continuous-time is undefined at 0

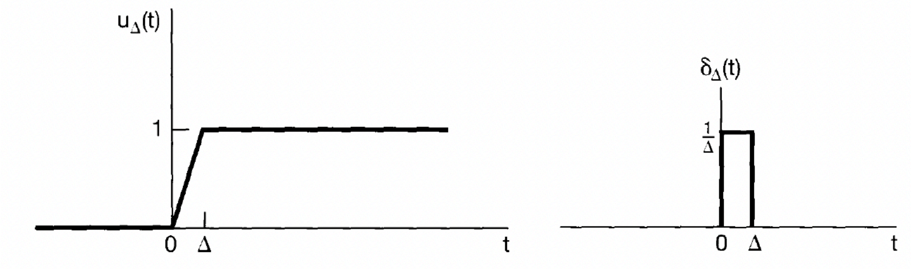
$$u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases}$$

The unit step is the **running integral** of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Conversely, the unit impulse function is considered the first derivative of the unit step

$$\delta(t) = \frac{du(t)}{dt}$$



2 Linear Time-Invariant Systems

2.5 Singularity Functions

- We introduce a set of related signals called the **singularity functions** to learn more about the idealized unit impulse function in continuous-time
- These signals are defined in terms of how they behave under convolution with other signals

2.5.1 The Unit Impulse as an Idealized Short Impulse

By the sifting property, the unit impulse can be seen as the impulse response of the identity system. We have for any $x(t)$,

$$x(t) = x(t) * \delta(t)$$

2.5.2 Defining the Unit Impulse Through Convolution

All properties of the unit impulse can be obtained through the operational definition given above. If we let $x(t) = 1$ for all t, then

$$1 = x(t) = \delta(t) * x(t) = \int_{-\infty}^{+\infty} \delta(\tau)x(t-\tau)\ d\tau = \int_{-\infty}^{+\infty} \delta(\tau)\ d\tau$$

With some more manipulation, we can see that

$$f(t)\delta(t) = f(0)\delta(t)$$

2.5.3 Unit Doublets and Other Singularity Functions

Consider the LTI system for which the output is the derivative of the input $y(t) = \frac{dx(t)}{dt}$

The unit impulse response of this system is the derivative of the unit impulse, called the unit doublet $u_1(t)$

$$\frac{dx(t)}{dt} = x(t) * u_1(t)$$

Taking the derivative again,

$$
\begin{aligned}
\frac{d^2x(t)}{dt^2} &= \frac{d}{dt}\frac{dx(t)}{dt} \\
&= x(t) * u_1(t) * u_1(t) \\
&= x(t) * u_2(t)
\end{aligned}
$$

We see that in general, we have

$$u_k(t) = u_1(t) * \cdots * u_1(t) \text{ (k times)}$$

Sometimes we use alternative notations for $\delta(t)$ and $u(t)$

$$
\begin{aligned}
\delta(t) &= u_0(t) \\
u(t) &= u_{-1}(t)
\end{aligned}
$$

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The Fourier Series

3 Fourier Series Representation of Periodic Signals

3.3 Fourier Series Representation of Continuous-Time Periodic Signals

3.3.1 Linear Combinations of Harmonically Related Complex Exponentials

- For a signal $\phi_k(t) = e^{j\omega_0 t}$ with a fundamental frequency ω_0, there is a set of **harmonically related** complex exponentials $\phi_k(t) = e^{jk\omega_0 t}$ for all integers k
    - The terms for $k = 1$ and $k = -1$ are called the **fundamental components** or **first harmonic components**
    - Components after $k = |1|$ are known as the Nth components

3.3.2 Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

Given

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

if we multiply both sides by $e^{-jn\omega_0 t}$ then integrate from 0 to T

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \left[\int_0^T e^{jk\omega_0 t} e^{-jn\omega_0 t}\, dt\right]$$

From MAT290 we can recall how to evaluate the integral in the brackets. Rewriting using Euler’s formula:

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j\int_0^T \sin(k-n)\omega_0 t\, dt$$

Thus,

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \begin{cases} T, k = n \\ 0, k \neq n \end{cases}$$

and

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\,dt$$

To summarize, if $x(t)$ can be expressed as a linear combination of harmonically related complex exponentials (or, if it has a Fourier series representation), then the coefficients are given by the equation above. This pair of equations defines the Fourier series of a periodic continuous-time signal.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{1}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt \tag{2}$$

3.4 Convergence of the Fourier Series

While Fourier maintained that any periodic signal can be represented by a Fourier series, this is not actually true. However, a Fourier series exists for all functions that we are concerned with in this course.

Let $x_N(t)$ be a finite series of the form

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

Let $e_N(t)$ denote the approximation error, represented by

$$e_N(t) = x(t) - x_N(t)$$

We need to specify a quantitative measure of how good any particular approximation is. We can use the criterion of the energy in the error over one period

$$E_N = \int_T |e_N(t)|^2\,dt$$

A set of conditions known as the **Dirichlet conditions** guarantees that $x(t)$ equals its Fourier series representation, except where $x(t)$ is discontinuous. These conditions will apply to all of the functions studied in this course.

1. Over any period, $x(t)$ must be absolutely integrable, that is

$$\int_T |x(t)|\,dt < \infty$$

2. In any finite interval of time, $x(t)$ is of bounded variation. There are no more than a finite number of maxima and minima during any single period of the signal.

3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

3.5 Properties of the Continuous-Time Fourier Series

Suppose that the function $x(t)$ is a periodic signal with period T and fundamental frequency ω_0 and the Fourier series coefficients of $x(t)$ are denoted by a_k.

3.5.1 Linearity

$$z(t) = Ax(t) + By(t) \text{ becomes } c_k = Aa_k + Bb_k$$

3.5.2 Time Shifting

The Fourier series coefficients b_k of the resulting signal $y(t) = x(t - t_0)$ may be expressed as

$$b_k = \frac{1}{T} \int_T x(t - t_0) e^{-jk\omega_0 t} \, dt$$

If we say that

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

then we have

$$x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t} \, a_k$$

The implication of this property is that when a periodic signal is shifted in time, the magnitude of its Fourier series coefficients does not change.

3.5.3 Time Reversal

If we say that

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

then we have

$$x(-t) \xleftrightarrow{\mathcal{FS}} a_{-k}$$

3.5.4 Time Scaling

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

3.5.5 Multiplication

$$x(t)y(t) \xleftrightarrow{\mathcal{FS}} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

3.5.6 Conjugation and Conjugate Symmetry

If we take the complex conjugate of a periodic signal $x(t)$, then we apply both complex conjugation and time reversal on the Fourier coefficients.

$$x^*(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}$$

3.5.7 Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

3.6 Fourier Series Representation of Discrete-Time Periodic Signals

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

A discrete-time signal is periodic if

$$x[n] = x[n+N]$$

The set of all discrete-time complex exponential signals periodic with N is given by

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, k = 0, \pm 1, \pm 2, \ldots$$

The summation as k varies of a range of successive N integers is

$$x[n] = \sum_{k=\langle N\rangle} a_k\phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

Determination of the Fourier Series Representation of a Periodic Signal

$$\sum_{n=\langle N\rangle} a_k e^{jk(2\pi/N)n} = \begin{cases} N, k = 0, \pm N, \pm 2N, \ldots \\ 0,\ \text{otherwise} \end{cases}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$$

3.7 Properties of Discrete-Time Fourier Series

$$x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

3.7.1 Multiplication

$$x[n]y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$$

3.7.2 First Difference

$$x[n] - x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} (1 - e^{-jk(2\pi/N)})a_k$$

3.7.3 Parseval's Relation for Discrete-Time Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2$$

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The Fourier Transform

4 The Continuous-Time Fourier Transform

4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Let's start by revisiting the Fourier series representation for the continuous-time periodic square wave. Over one period,

$$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, T_1 < |t| < T/2 \end{cases}$$

and periodically repeats with period T. The Fourier series coefficients a_k are given by:

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

where $\omega_0 = 2\pi/T$.

We consider an aperiodic signal as the limit of a periodic signal as the period becomes large, and we examine the Fourier representation of that signal. For some number T_1, $x(t) = 0$ is $|t| > T_1$. From this aperiodic signal, we construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period. As we select a longer period, $\tilde{x}(t)$ becomes identical to $x(t)$.

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\, dt$$

By defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-jk\omega t}\, dt$$

we have the coefficients a_k

$$a_k = \frac{1}{T}X(jk\omega_0)$$

By combining the above summation and the second formula for the coefficients, we get

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

Ultimately, we have the equations that give us the Fourier transform pair, with $X(j\omega)$ referred to as the **Fourier Transform** or **Fourier integral** of $x(t)$ and $x(t)$ gives us the **inverse Fourier transform**, or **synthesis** equation.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{jwt} \, d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} \, dt$$

- For periodic signals, these complex exponentials have amplitudes $\{a_k\}$ which occur at a discrete set of harmonically related frequencies $k\omega_0$.
- For aperiodic signals, the complex exponentials occur at a continuum of frequencies.
  - These frequencies have an ‘amplitude’ given by $X(j\omega(d\omega/2\pi)$
  - The transform $X(j\omega)$ of an aperiodic signal is referred to as the spectrum
    - This is because the transform gives us the information we require to reconstruct the signal as a linear combination of sinusoidal signals at different frequencies

$$a_k = \frac{1}{T} X(j\omega) \Big|_{\omega = k\omega_0}$$

4.1.2 Convergence of Fourier Transforms

- To derive the Fourier equations above, we assume that $x(t)$ is of a finite duration, however, the equations are valid for many infinite-duration signals too.
  - Our derivation of the Fourier transform suggests we can apply the same definition and criteria for convergence.
- Just as with periodic signals, there is a set of conditions (the **Dirichlet conditions**) that guarantee that $\tilde{x}(t)$ is equal to $x(t)$ for any t except for at a discontinuity.
  - Where there is a discontinuity, the value at that t is the is average value on either side of the discontinuity.

The Dirichlet conditions require that:

1. $x(t)$ is absolutely integrable
2. $x(t)$ have a finite number of maxima and minima within any finite interval
3. $x(t)$ have a finite number of discontinuities within any finite interval. Each discontinuity must be finite.

4.3 Properties of the Continuous-Time Fourier Transform

Sometimes we will refer to $X(j\omega)$ with the notation $\mathcal{F}\{x(t)\}$ and similarly $x(t)$ with the notation $\mathcal{F}^{-1}\{X(j\omega)\}$. We will use this notation to refer to a Fourier transform pair:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

4.3.1 Linearity

$$ax(t) + by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(j\omega) + bX(j\omega)$$

4.3.2 Time Shifting

$$x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

4.3.3 Conjugation and Conjugate Symmetry

$$x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X(-j\omega)$$

4.3.4 Differentiation and Integration

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\, dt \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

4.3.5 Time and Frequency Scaling

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X(j\omega/a)$$

$$x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-j\omega)$$

4.3.6 Duality

$$-jt\, x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}$$

$$e^{j\omega_0 t} x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

$$-\frac{1}{jt} x(t) + \pi x(0)\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\tau)\, d\tau$$

4.3.7 Parseval's Relation

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega$$

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5 The Discrete-Time Fourier Transform

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

5.1.1 Development and the Discrete-Time Fourier Transform

- We know that Fourier series coefficients for a continuous-time periodic square wave can be viewed as samples of an envelope function.
    - As the period of the square wave increases, these samples become more finely spaced.
    - Suggestion in chapter 4: represent an aperiodic signal $x(t)$ by constructing a periodic signal $\tilde{x}(t)$ that equals $x(t)$ over one period.
        - As this period approaches infinity $\tilde{x}(t)$ was equal to $x(t)$ over larger and larger intervals of time.

The discrete-time Fourier transform is given by the following equations:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

We see that the discrete-time Fourier transform shares many similarities with the continuous-time case. The major differences between the two are the periodicity of the DT transform and the finite interval of interrogation in the synthesis equation.

5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

Conditions on $x[n]$ that guarantee the convergence of this sum are direct counterparts of the convergence conditions for the CT Fourier transform. The first DT Fourier transform equation will converge if $x[n]$ is absolutely summable.

5.3 Properties of the Discrete-Time Fourier Transform

We will use the following to represent the DT Fourier transform

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

5.3.1 Periodicity of the DT Fourier Transform

$$X(e^{j(\omega + 2\pi)}) = X(e^{j\omega})$$

5.3.2 Linearity of the Fourier Transform

$$ax_1[n] + bx_2[n] \overset{\mathcal{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

5.3.3 Time Shifting and Frequency Shifting

$$x[n - n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

5.3.4 Conjugation and Conjugate Symmetry

$$x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega})$$

5.3.5 Differencing and Accumulation

$$x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega})$$

5.3.6 Time Reveral

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$$

5.3.7 Time Expansion

$$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jk\omega})$$

5.3.8 Differentiation in Frequency

$$nx[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

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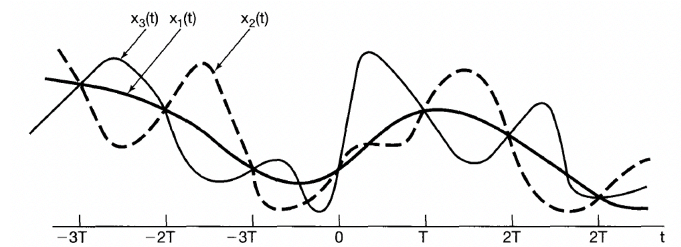
Sampling, Aliasing, and Interpolation

7 Sampling

7.1 Representation of a Continuous-Time Signals: The Sampling Theorem

Generally we would not expect that a signal could be uniquely specified by a sequence of equally spaced samples in the absence of additional conditions or information. However, if a signal is **band limited** (zero Fourier transform outside a finite band of frequencies) and if the samples are taken sufficiently close to the highest frequency, then the samples uniquely specify the signal and it **can be reconstructed perfectly**.

This result is the **sampling theorem**.



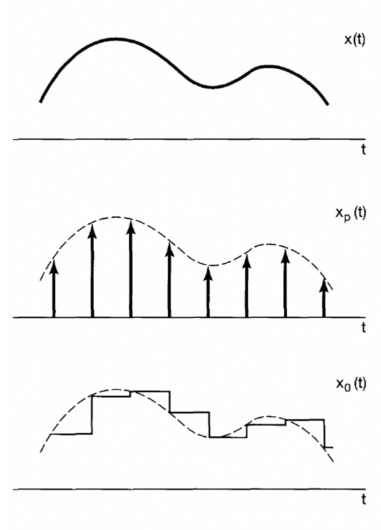
7.1.1 Impulse-Train Sampling

- We need a convenient way in which to represent the sampling of a CTS.
- We can do this by using a periodic impulse train multiplied by the CTS
  - This mechanism is called impulse-train sampling
  - Periodic impulse train $p(t)$ is called the sampling function
  - T is the sampling period
  - $\omega_s/2\pi/T$ is the sampling frequency

$$x_p(t) = x(t)p(t) \text{ where } p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

7.1.2 Sampling with a Zero-Order Hold

- In this system, a CTS $x(t)$ is sampled at a given instant, and its value is held until the next sample is taken.



7.2 Reconstruction of a Signal From its Samples Using Interpolation

- Fitting os a continuous signal to a set of example values is a commonly used procedure for reconstructing a function.
- One example is the zero-order hold discussed previously.
- Another example is to connect the sample points with a straight line.
- Interpolation using the impulse response of an ideal lowpass filter is also called band-limited interpolation, since it implements exact reconstruction of a CTS $x(t)$ if it's band limited

7.3 The Effect of Undersampling: Aliasing

- When $\omega_s < 2\omega_M$, the spectrum of $x(t)$ is no longer replicated and is thus no longer recoverable by lowpass filtering.
- When aliasing occurs, the original frequency takes on the identity of a lower frequency $\omega_s - \omega_0$
- The sampling theorem explicitly requires that the sampling frequency be greater than twice the highest frequency in the signal.

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Fundamentals of Continuous-Time Signals

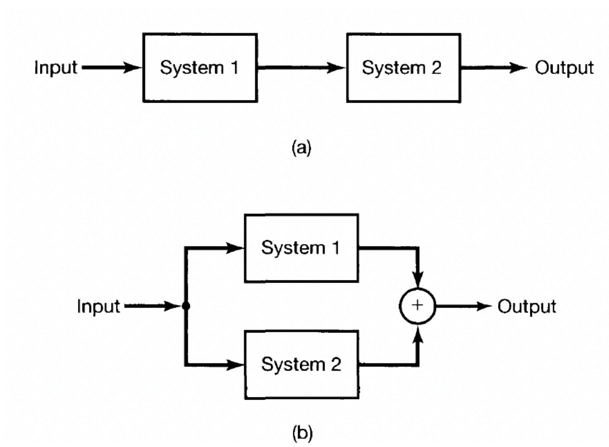
1 Signals and Systems

1.5 Continuous-Time and Discrete-Time Systems

- Systems are interconnection of components, devices, or subsystems.
- A continuous-time system is one in which CTS are applied and result in CTS outputs.
- we identify classes of systems with two important characteristics
  - The systems in this class have properties and structures we exploit to gain insight into their behaviour and to develop effective tools for their analysis
  - Many systems of practical importance can be accurately modelled using systems in this class.

Interconnections of Systems

Series and parallel interconnections



We use the notation

$$x(t) \to y(t)$$
$$x[n] \to y[n]$$

1.6 Basic System Properties

Systems With and Without Memory

- A system is memoryless if its output for each value is dependent only on the input value at the same time.

- In many systems, memory is directly associated with the storage of energy

Invertibility and Inverse Systems

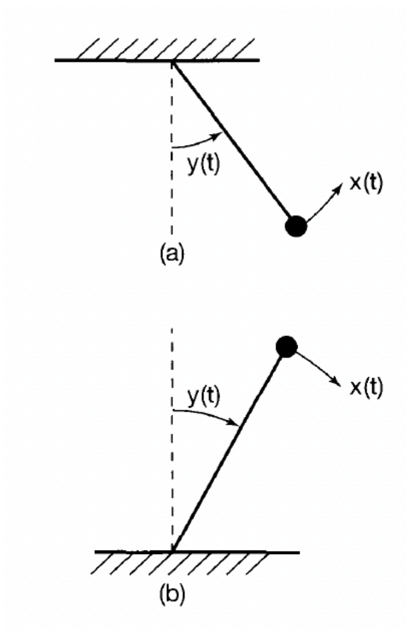
- A system is invertible if distinct inputs lead to distinct outputs

Causality

- A system is causal id the output at any given time depends only on values of the input at present time and past
  - Also called nonanticipative as the system does not anticipate future values of the input.
  - All memoryless systems are causal

Stability

- A stable system is one in which small inputs lead to responses that do not diverge



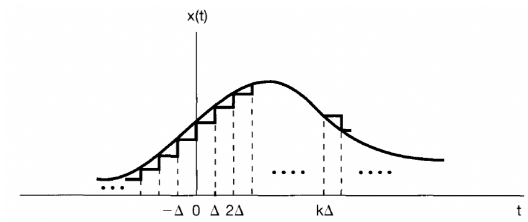
Time Invariance

- A system is time invariant if its behaviour and characteristics are fixed over time.
- For example, consider an RC circuit. If we run the circuit today, then again tomorrow, it will behave the same

2 Linear Time-Invariant Systems

2.2 Continuous-Time LTI Systems: The Convolution Integral

- To develop CT counterpart of the discrete sifting property, we begin by considering a pulse approximation



If we define

$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, 0 \leq t \leq \Delta \\ 0, \text{ otherwise} \end{cases}$$

then $\Delta\delta_{\Delta}(t)$ has unit amplitude and we can write a summation similar to DT case. As Δ approaches zero, the approximation becomes better and the limit becomes $x(t)$.

Thus,

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)\, d\tau$$

This is the sifting property of the continuous-time impulse. We also use the notation

$$y(t) = x(t) * h(t)$$

2.3 Properties of Linear Time-Invariant Systems

2.3.1 The Commutative Property

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

in discrete time, and in continuous time

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\, d\tau$$

2.3.2 The Distributive Property

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

- the above simply states that the response of an LTI system to the sum of two inputs must equal the sum of the responses to these signals individually

2.3.3 The Associative Property

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

2.3.4 LTI Systems With and Without Memory

- a system is memoryless if its output at any time depends only on the value of the input at that same time
- if a discrete-time LTI system has an impulse response $h[n]$ not identically zero for $n \neq 0$, then the system has memory

2.3.5 Invertibility of LTI System

$$h[n] * h_1[n] = \delta[n]$$

2.3.6 Causality for LTI Systems

$$h[n] = 0 \text{ for } n < 0$$

2.3.7 Stability for LTI Systems

- if the impulse response is absolutely summable (absolutely integrable in the continuous case) or has finite action, then we call it BIBO stable
  - BIBO = bounded-input-bounded-output

2.3.8 The Unit Step Response of an LTI System

- the step response of a discrete-time LTI system is the convolution of the unit step with the impulse response

$$s[n] = u[n] * h[n]$$

- the unit impulse response if the first derivative of the unit step response
-

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

2.4 Causal LTI Systems Described by Differential and Difference Equations

- an important class of continuous-time systems is that for which the input and output are related through a **linear constant-coefficient differential equation (LICC-ODE)**

2.4.1 Linear Constant-Coefficient Differential Equations

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

- DEs provide an implicit specification of the system, they describe a relationship between the input and output rather than an explicit expression
- we emphasize that the condition of initial rest does not specify a zero initial condition at a fixed point in time, but rather adjusts this point in time so that the response is zero until the input becomes nonzero

2.4.2 Linear Constant-Coefficient Difference Equations

- counterpart of the constant-coefficient derivative is the linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

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Analysis of Continuous-Time Signals Using the Fourier Transform

3 Fourier Series Representation of Periodic Signals

3.3 Fourier Series Representation of Continuous-Time Periodic Signals

3.3.1 Linear Combinations of Harmonically Related Complex Exponentials

- in chapter 1, we introduced the two basic periodic signals, the sinusoid $x(t) = \cos \omega t$ and the complex exponential $x(t) = e^{j\omega_0 t}$
    - both signals are periodic with a fundamental frequency and period, and have a set of harmonically related complex exponentials

3.3.2 Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

- orthogonality

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \begin{cases} T, \ k = n \\ 0, \ k \neq n \end{cases}$$

6 Time and Frequency Characterization of Signals and Systems

6.2 The Magnitude-Phase Representation of the Frequency Response of LTI Systems

- when the magnitude or phase of the input signal is modified in an unwanted way, we call it a distortion

6.2.1 Linear and Nonlinear Phase

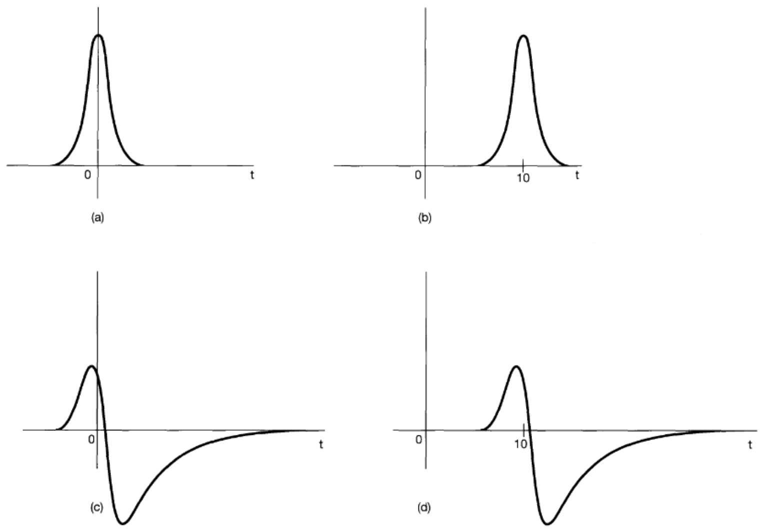

Figure 6.3 (a) Continuous-time signal that is applied as the input to several systems for which the frequency response has unity magnitude; (b) response for a system with linear phase; (c) response for a system with nonlinear phase; and (d) response for a system with phase equal to the nonlinear phase of the system in part (c) plus a linear phase term.

- systems where magnitude is unchanged (unity-gain) is called an **all-pass system**

6.2.2 Group Delay

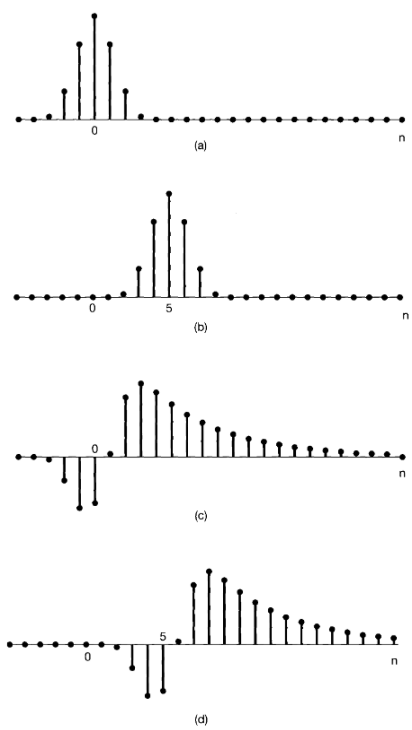

Figure 6.4 (a) Discrete-time signal that is applied as input to several systems for which the frequency response has unity magnitude; (b) response for a system with linear phase with slope of −5; (c) response for a system with nonlinear phase; and (d) response for a system whose phase characteristic is that of part (c) plus a linear phase term with integer slope.

6.2.3 Log-Magnitude and Bode Plots

$$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$$

Analysis of Continuous-Time Signals Using the Laplace Transform

9 The Laplace Transform

9.1 The Laplace Transform

- When we replace the $j\omega$ complex exponential variable with a general variable s, we have the Laplace transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

- THe range of values of s for which the Laplace integral converges is called the region of convergence
- If we express $X(s)$ as a rational function with numerator $N(s)$ and denominator $D(s)$, then we say:
    - the roots of the numerator are called the zeroes of the function
    - the roots of the denominator are called the poles of the function

9.2 The Region of Convergence for Laplace Transforms

- There are some specific constraints on the ROC for various classes of signals
- Property 1: the ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis of the s-plane.
- Property 2: for rational Laplace transforms, the ROC does not contain any poles.
- Property 3: if $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane
- Property 4: if $x(t)$ is right-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC.
- Property 5: if $x(t)$ is left-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} < \sigma_0$ will also be in the ROC.
- Property 6: If $x(t)$ is two-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of the strip in the s-place that includes the line $Re\{s\} = \sigma_0$
- Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

9.3 The Inverse Laplace Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(s) e^{st} \, d\omega$$

9.5 Properties of the Laplace Transform

9.5.1 Linearity of the Laplace Transform

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$$

9.5.2 Time Shifting

$$x(t - t_0) \longleftrightarrow e^{-st_0} X(s)$$

9.5.3 Shifting in the s-Domain

$$e^{s_0 t} x(t) \longleftrightarrow X(s - s_0)$$

9.5.4 Time Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|} X(s/a)$$

9.5.5 Conjugation

$$x^*(t) \longleftrightarrow X^*(s^*)$$

9.5.6 Convolution Property

$$x_1(t) * x_s(t) \longleftrightarrow X_1(s) X_2(s)$$

9.5.7 Differentiation in the Time Domain

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s)$$

9.5.8 Differentiation in the s-Domain

$$-tx(t) \longleftrightarrow \frac{dX(s)}{ds}$$

9.9 The Unilateral Laplace Transform

- We now introduce the unilateral Laplace transform

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} \, dt$$

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Fundamentals of Discrete-Time Signals

10 The z-Transform

10.8 System Function Algebra and Block Diagram Representation

- The z-transform in discrete time allows us to replace time-domain operations with algebraic operations