MAT290 Advanced Engineering Mathematics

1. Intro to DEs

- 1.2 Initial Value Problems
  - **THEOREM 1.2.1 Existence of a Unique Solution**

    Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$, that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$ defined on I_0 that is a solution of the initial-value problem (2).

    Existence and Uniqueness

2. First-Order DEs

- 2.1 Solution Curves w/o a Solution
  - Direction and slope fields
  - Autonomous DEs - free of independent variable
  - Stable, semi-stable, and unstable critical points
  - Translation property - adidng constant to solution still makes a solution
- 2.3 Linear Equations
  - Standard form of DE
  - Integrating Factor method - for first order autonomous ODEs
  - Singular points
  - $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ and $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$,

    erf(x)+erfc(x)=1

3. Higher-Order DEs

- 3.1 Theory of Linear Equations
  - **THEOREM 3.1.1 Existence of a Unique Solution**

    Let $a_n(x)$, $a_{n-1}(x)$, …, $a_1(x)$, $a_0(x)$, and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

    Existence and uniqueness - for an n-th order derivative, n-th term must be continuous and non-zero, and n-1 to 0 must continuous
  - In order to solve non-homogeneous, we need to first solve homogeneous
  - Superposition applies to solutions of homogeneous DEs
  - Linear Dependence and Linear Independence
  - $W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$

    Wronskian determinant - If det is non-zero, then the set of solutions is linearly independent
  - Any set of independent solutions to homogeneous linear DE is called **fundamental set**.
- 3.2 Reduction of Order
  - For homogeneous linear 2nd order ODEs where we are given one solution
  - $y_2(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}dx$

    y1(x) != 0 for all x
  - Use Wronskian to prove linear independence
- 3.3 Linear Equations w/ Constant Coefficients
  - Auxiliary Equation
    - Case I: Distinct Real Roots
    - Case II: Repeated Real Roots
    - Case III: Complex Conjugate Roots
  - Auxiliary equation also applies for higher-order differential equations
- 3.5 Variation of Parameters
  - Find complementary solution to corresponding homogeneous ODE
  - $u_1(x) = -\int_{x_0}^x \frac{y_2(t)f(t)}{W(t)}dt$ and $u_2(x) = \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)}dt$

    yp(x) = u1(x)y1(x)+u2(x)y2(x)
- 3.8 Linear Models: Initial-Value Problems
  - Spring mass systems with undamped and damped motion
  - $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

    Circuit questions

4. Laplace Transforms

- 4.1 Definition of the Laplace Transform
  - $F(s) = \int_0^\infty e^{-st}f(t)dt$

    Equation of Laplace transform
  - Sufficient conditions - f must be of exponential order
    |f(t)| <= Me^(ct) for all t > T
- 4.2 Inverse Transforms and Transforms of Derivatives
  - Laplace is a linear transform
  - Solving Linear ODEs w/ the Laplace
- 4.3 Translation Theorems
  - First Translation Theorem
    - $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
    - $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$
  - Unit-step or Heaviside function U(t-T)
  - Second Translation Theorem
    - $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$
    - $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$
- 4.4 Additional Operational Properties
  - Derivatives of Transforms
    - $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$
  - Convolution
    - $f * g = \int_0^t f(\tau)g(t-\tau)d\tau$
    - $\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$
    - $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$
  - Periodic Function
    - $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)dt$
- 4.5 Delta-Dirac Function
  - $\delta_a(t-t_0) = \begin{cases} 0, 0, \le t < t_0 - a \\ \frac{1}{2a}, t_0 - 1 \le t < t_0 + a \\ t \ge t_0 + a \end{cases}$

    Laplace of Delta function is 1
  - $\int_0^\infty f(t)\delta(t-t_0)dt = f(t_0)$

    Sifting property

17. Complex Functions & Variables

- 17.2 Powers and Roots
  - z=r(cos(t) + isin(t))
  - r = |z|
    theta = arg(z)
  - principal argument (-pi, pi]
  - De Moivre's Formula
    (cos(t)+isin(t))^n=cos(nt)+isin(nt)
  - Finding roots of a complex number
- 17.3 Sets in the Complex Plane
  - We can think of complex sets as a disk within a radius
  - **Interior** and **boundary** points
  - If any two points can be connected by polygon line entirely within the set, then it's **connected**
  - A region containing all of its boundary points is **closed**. containing none of its boundary points is **open**.
- 17.4 Functions of a Complex Variable
  - Function continuous at a point if limit as z goes to z0 of f(z) is f(z0)
  - $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

    Limit definition of a complex derivative
- 17.5 Cauchy-Riemann Equations
  - $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

    Cauchy-Riemann Equations
  - A real-valued function f(x,y) that has continuous and satisfies the Laplace equation is called **harmonic**.
    - $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$

      Laplace equation
  - How to find harmonic conjugate functions?
  - If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v are harmonic in D. Sometimes, it is possible to find another function $v(x,y)$ that is harmonic in D so that $u(x,y) + iv(x,y)$ is harmonic in D. Function v is then called a harmonic conjugate function of u.

    Harmonic Conjugate Functions
- 17.6 Exponential and Logarithmic Functions
  - ln(z)=log_e|z|+i(theta+2kpi)
- 17.7 Trigonometric and Hyperbolic Functions
  - expansion of sin, cos, sinh, and cosh
  - sin = sin*cosh+icos*sinh
    cos=cos*cosh-isin*sinh

18. Complex Calculus

- 18.1 Contour Integrals
  - $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$

    Formula for evaluating a complex integral
  - $\left(\oint_C f \cdot \mathbf{T}ds\right) + i\left(\oint_C f \cdot \mathbf{n}ds\right) = \oint_C (u - iv)(dx + idy) = \oint \bar{f(z)}dz$

    Real integral is the **circulation** and imaginary integral is the **net flux**
- 18.2 Cauchy-Goursat Theorem
  - Simply and multiply connected domains
  - **Cauchy-Goursat Theorem**
    Suppose a function f is analytic in a simply connected domain. Then for every closed contour C, we have oint f(z)dz=0
- 18.3 Independence of Path
  - If f is an analytic function in a simply connected domain, then int f(z)dz is independent of the path C
  - $\int_C f(z)dz = F(z_1) - F(z_0)$

    Fundamental Theorem for Contour Integrals
- 18.4 Cauchy's Integral Formula
  - $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz$

    f is analytic in a simply connected domain and C is a simple closed contour lying entirely within D. z0 must be a point within C
  - $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$

    Cauchy's Integral Formula for derivatives
  - **Liouville's Theorem**
    The only bounded entire functions are constants
  - **Fundamental Theorem of Algebra**
    If P(z) is a nonconstant polynomial, then P(z)=0 has at least one root

19. Series and Residues

- 19.1 Sequences and Series
  - **Criterion for Convergence**
    A sequence {z_n} converges to a complex number L iff Re(zn) converges to Re(L) and Im(zn) converges to Im(zn)
  - **Absolute Convergence**
    An infinite series is said to be absolutely convergent if the equivalent series with the sum term in absolute brackets also converges
  - Tests for convergence:
    1. Ratio test
    2. Root test
  - Circle of convergence can be 0, a finite number, or infinity
- 19.2 Taylor Series
  - Term-by-term integration and differentiation
  - $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$

    Taylor's Theorem
- 19.3 Laurent Series
  - $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$

    Laurent Series
    Negative power part of the series is called the **principal** part and the positive power part is called the **analytic** part
    - $a_k = \frac{1}{2\pi i}\oint_C \frac{f(s)}{(s - z_0)^{k+1}}ds$

      How to find coefficients of the Laurent Series
- 19.4 Zeros and Poles
  - Singular point of a complex function can be classified on number of terms in principal part of Laurent Series
    - If no terms in principal part, or all parts are zero, the singularity is called a **removable singularity**
    - If principal part contains finite number of terms (say n terms), then the point is a **pole of order n**. If pole is of order 1, then we call it a **simple pole**.
    - If principal part contains infinitely many terms, we call it an **essential singularity**
- 19.5 Residue Theorem
  - $a_{-1} = \mathrm{Res}(f(z), z_0)$

    Coefficient of the last term in the Laurent Series is called the **residue**
  - $\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!}\lim_{z \to z_0}\frac{d^{n-1}}{dz^{n-1}}(z - z_0)^n f(z)$

    Residue at a Pole of Order n
    - $\mathrm{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$

      Alternative formula for calculating **residue at a simple pole**
  - $\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$

    Cauchy's Residue Theorem
    Let D be a simply connected domain and C a simple contour lying entirely within D. If a function f is analytic on and inside C, except at a finite number of singular points within C
- 19.6 Evaluation of Real Integrals
  - $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

    Circles centered at the origin
  - $\int_{-\infty}^{\infty} f(x)dx$

    Defined from -∞ to r and r to ∞
  - $\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$

    cos and sin
  - $\lim_{r \to 0}\int_{C_r} f(z)dz = \pi i\,\mathrm{Res}(f(z), c)$

    Indented Contour Method
  - Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \ge n + 1$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \le \theta \le \pi$, and $\alpha > 0$, then $\int_{C_R} (P(z)/Q(z))e^{i\alpha z}\,dz \to 0$ as $R \to \infty$.

    Jordan's Lemma - Big semicircle contour around the top goes to zero for a function f = P/Q if Q is at least one degree higher than P.

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