MIE100: Dynamics

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Contents

1 Kinematics of Particles

Rectilinear Motion

$$v = \frac{ds}{dt} = \dot{s} \tag{1}$$

$$d = \frac{dv}{dt} = \ddot{s} \tag{2}$$

- If we eliminate dt, we get $v\ dv = a\ ds$.

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a \cdot dt \tag{3}$$

$$\int_{v_1}^{v_2} v \cdot dv = \int_{s_1}^{s_2} a \cdot ds \textbf{ OR } \frac{1}{2}(v_2^2 - v_1^2) \tag{4}$$

Plane Curvilinear Motion

$$v = \lim_{t \to 0} \frac{\Delta r}{\Delta t} \to v = \frac{dr}{dt} = \dot{r} \tag{5}$$

For projectile motion, $a_x = 0$ and $a_y = -g$.

Normal and Tangential (n-t) Coordinates

$$v = v\hat{e_t} = \rho\dot{\beta}\hat{e_t} \tag{6}$$

where ρ is the radius of curvature and β is in radians.

For circular motion,

$$a = \frac{v^2}{r} e_n + \dot{v} e_t \tag{7}$$

$$v = r\dot{\theta}, a_n = \frac{v^2}{r} = r\dot{\theta}^2 = v\dot{\theta}, a_t = \dot{v} = r\ddot{\theta} \tag{8}$$

Polar Coordinates $(r - \theta)$

Velocity:

$$v = \dot{r}\hat{e_r} + r\dot{\theta}\hat{e_\theta} \tag{9}$$

Acceleration:

$$a = (\ddot{r} - r\dot{\theta}^2)\hat{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e_\theta} \tag{10}$$

For circular motion with a constant r, components become:

- $v_r = 0$
- $v_\theta = r\dot{\theta}$
- $a_r = -r\dot{\theta}^2$
- $a_\theta = r\ddot{\theta}$

Relative Motion (Translating Axis)

$$r_B = r_A + r_{B/A} \rightarrow v_B = v_A + v_{B/A} \rightarrow a_B = a_A + a_{B/A} \tag{11}$$

2 Kinetics of Particles

Newton's Second Law

$$F = ma \tag{12}$$

Equation of Motion and Solutions of Problems

$$\sum F = ma \tag{13}$$

Remember to determine if motion is constrained or unconstrained.

Rectilinear Motion

Split up $\sum F = ma$ into its directional components.

Curvilinear Motion

$$\sum F_r = ma_r \rightarrow a_r = \ddot{r} - r\dot{\theta}^2 \tag{14}$$

$$\sum F_\theta = ma_\theta \rightarrow a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{15}$$

Work and Kinetic Energy

$$U = \int F \cdot dr \tag{16}$$

1. Work associated with a constant external force,
2. Work associated with a spring force, and
3. Work associated with weight.

$$U_{1\rightarrow 2} = \int_1^2 F \cdot dr = \int_{v_1}^{v_2} mv \cdot dv = \frac{1}{2}m(v_2^2 - v_1^2) \tag{17}$$

$$E_K = \frac{1}{2}mv^2 \tag{18}$$

$$U_{1\rightarrow 2} = T_2 - T_1 = \Delta T \tag{19}$$

Power: $P = F \cdot v$.

Potential Energy

$$V_g = mgh \tag{20}$$

$$\Delta V_e = \frac{1}{2}k(x_2^2 - x_1^2) \tag{21}$$

Work-energy equation: $T_1 + V_1 + U_{1\to 2} = T_2 + V_2$

Linear Impulse and Linear Momentum

$$\sum F = m\dot{v} \to \sum F = \dot{G} \tag{22}$$

Impulse:

$$\int_{t_1}^{t_2} \sum F \cdot dt = G_2 - G_1 = \Delta G \to G_1 + \int_{t_1}^{t_2} \sum F \cdot dt = G_2 \tag{23}$$

Angular Impulse and Angular Momentum

$$H_O = r \times mv \to \sum M_O = \dot{H}_O \tag{24}$$

3 Kinetics of Systems of Particles

Generalized Newton's Second Law

For a system of particles: $mr = \sum m_i r_i$ Kinetic energy expression: $v_i = v + \dot{\rho}_i$, where ρ_i is the velocity of m_i with regards to a translating reference frame moving with the mass centre G.

$$T = \frac{1}{2}mv^2 + \sum \frac{1}{2}m_i|\dot{\rho}_i| \tag{25}$$

Impulse-Momentum

$$G = mv \to \sum F = \dot{G} \tag{26}$$

Angular momentum:

- About a fixed point O:

$$H_O = \sum(r_i \times m_i v_i) \tag{27}$$

$$\sum m_O = \dot{H}_O \tag{28}$$

- About the mas centre G:

$$H_G = \sum \rho_i \times m_i \dot{r}_i \tag{29}$$

$$\sum M_G = \dot{H}_G \tag{30}$$

- About an arbitrary point P:

$$H_P = H_G + \rho \times mv \tag{31}$$

$$\sum M_P = \dot{H}_G + \rho \times ma \tag{32}$$

Conservation of Energy and Momentum

$$\Delta T + \Delta V = 0 \rightarrow T_1 + V_1 = T_2 + V_2 \tag{33}$$

$$(H_O)_1 = (H_O)_2 \textbf{ OR } (H_G)_1 = (H_G)_2 \tag{34}$$

4 Plane Kinematics of Rigid Bodies

Rotations

$$\omega = \omega_0 + \alpha t \tag{35}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \tag{36}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{37}$$

Rotation about a fixed axis:

- $v = \omega \times r$
- $a_n = \omega \times \omega \times r = \frac{v^2}{r} = \omega \times v$
- $a_t = \alpha \times r$

Relative Velocity

SINCLAIR'S SPAM EQUATION: $v_B = v_A + \omega_{AB} \times r_{B/A}$

$$\Delta r_A = \Delta r_B + \Delta r_{A/B} \tag{38}$$

Instantaneous Centre of Zero Velocity

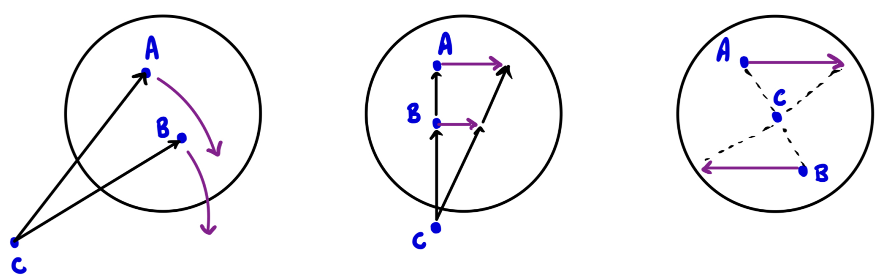

Figure 1: Instantaneous Centre of Zero Velocity

Relative Acceleration

$$a_A = a_B + a_{A/B} \tag{39}$$

$$(a_{A/B})_n = \frac{v_{A/B}^2}{r} = r\omega^2 \tag{40}$$

$$(a_{A/B})_t = \dot{v_{A/B}} = r\alpha \tag{41}$$

5 Plane Kinetics of Rigid Bodies

General Equations of Motion

$$\sum F = ma \rightarrow \sum M_G = \dot{H}_G \tag{42}$$

$$H_G = I\omega \rightarrow \sum M_G = \dot{H}_G = I\dot{\omega} = I\alpha \tag{43}$$

$$\sum M_P = \dot{H}_G + \rho \times ma \tag{44}$$

This is summed up as: $M_P = I\alpha + mad$

$$\sum M_O = I_O\alpha \tag{45}$$

Analysis procedure:

1. Kinematics – equations
2. Diagrams – identify knowns and unknowns
3. Equations of motion – use to get extra variables and solvable system

Translation

For a translating body, our general equations of motion are:

$$\sum F = ma \tag{46}$$

$$\sum M_G = I\alpha = 0 \tag{47}$$

Fixed-Axis Rotation

Almost same equations are applicable here:

$$\sum F = ma \tag{48}$$

$$\sum M_G = I\alpha \tag{49}$$

$$\sum M_O = I_O\alpha \tag{50}$$

General Plane Motion

Solving plane motion problems:

1. Choice of coordinate system
2. Choice of moment equations – $\sum M_P = I\alpha + mad$
3. Choice of constrained vs unconstrained motion
4. Number of unknowns
5. Identify body or system
6. Kinematics equations
7. Consistency in assumptions

Work-Energy Relations

$$U = \int F \cdot dr \textbf{ OR } U = \int (F \cos\alpha) ds \tag{51}$$

Kinetic energy:

- Translation – $T = \frac{1}{2}mv^2$
- Fixed axis rotation – $T = \frac{1}{2}I_O\omega^2$
- General plane motion – $T = \frac{1}{2}mv^2 + \frac{1}{2}I_O\omega^2$
  - Can also be expressed at the IC – $T = \frac{1}{2}I_C\omega^2$

6 Vibration & Time Response

Free Vibration of Particles

Applying Newton's Second Law in the form $\sum F_x = m\ddot{x}$:

- $-kx = m\ddot{x}$ **OR** $m\ddot{x} + kx = 0$
- Oscillation of a mass objected to a linear restoring force as described by this equation is called SIMPLE HARMONIC MOTION and is characterized by acceleration which is proportional to the displacement but of the opposite sign.
- $\ddot{x} + \omega_n^2 x = 0$, which gives us $\omega_n = \sqrt{k/m}$.

  Solution for undamped free motion:

$$x(t) = A\cos\omega_n t + B\sin\omega_n t \tag{52}$$

$$x(t) = C\sin(\omega_n t + \psi) \tag{53}$$

At $t = 0$, we get $x_0 = A$, and $\dot{x}_0 = B\omega_n$. If $t = 0$, then $x_0 = C\sin\psi$ and $\dot{x}_0 = C\omega_n\cos\psi$.

- $\psi = \tan^{-1}(\frac{x_0\omega_n}{\dot{x}_0}$

Natural frequency: $f_n = \frac{\omega_n}{2\pi}$

For damped free vibration:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{54}$$

If we define $\zeta = \frac{c}{2m\omega_n}$, then we can say:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{55}$$

For free damped vibration:

$$x = Ae^{rt} \tag{56}$$

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{57}$$

$$x = A_1e^{r_1t} + A_2e^{r_2t} \tag{58}$$

Categories of damped motion:

- $\zeta > 1$ OVERDAMPED
- $\zeta = 1$ CRITICALLY DAMPED
- $\zeta < 1$ UNDERDAMPED

Forced Vibration of Particles

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0 \sin\omega t}{m} \tag{59}$$

Vibration of Rigid Bodies

Use same equations derived throughout the course, just add rotation to the calculations.