MAT187: Calculus II

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Contents

1 Methods of Integration

1.1 Integration By Parts

From the product rule to integration by parts.

$$(u(x) \cdot (v(x))' = u'(x) \cdot v(x) + u(x) \cdot v'(x) \tag{1}$$

$$(uv)' = u'v + uv' \rightarrow uv' = (uv)' - u'v \tag{2}$$

$$\int u(x) \cdot v'(x) \cdot dx = u(x) \cdot u(x) - \int v(x) \cdot u'(x) \cdot dx \tag{3}$$

1.2 Trigonometric Substitution

Recall that $\cos^2(\theta) + \sin^2(\theta) = 1$ and $\tan^2(\theta) + 1 = \sec^2(\theta)$.

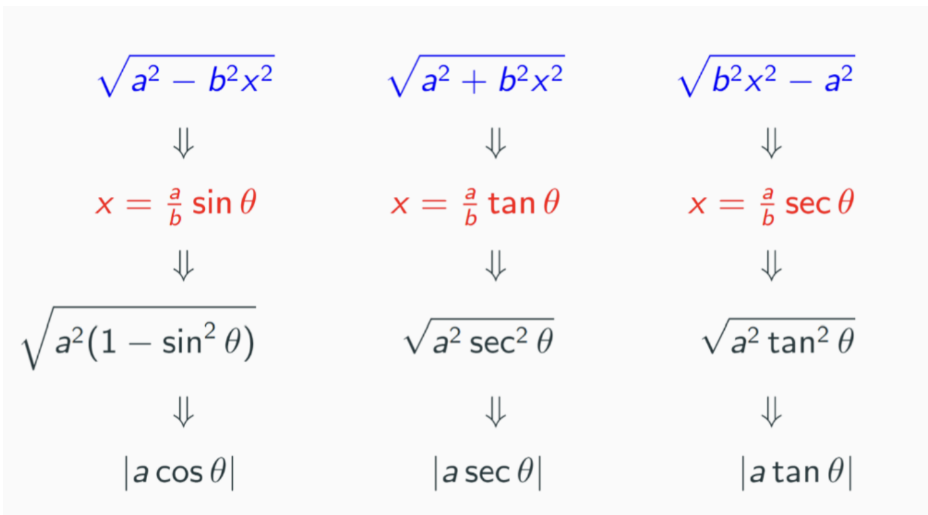

Figure 1: There Are Three General Types of Trig Sub Problems

1.3 Partial Fractions

Degree of the numerator has to be one less than the denominator for Heaviside's Rule to work. Use polynomial long division if it isn't.

$$\int \frac{3x}{x^2 - x - 2}dx = \int \frac{1}{x+1} + \frac{2}{x-2}dx = \ln|x+1| + 2\ln|x-2| + C \tag{4}$$

1.4 Approximating Integrals and Estimating Error

The Trapezoid Rule is NOT the same thing as a Riemann Sum. The Midpoint Rule is essentially the average of the left and right Riemann Sums.

1.5 Improper Integrals with Infinite Bounds

Three-step process to solving most math problems:

1. Find a way to approximate the answer
2. Find a method to improve the approximation

3. Take the limit of that method to infinity

Consider e^{-x} as a finite integral from $[0, \infty)$

$$\lim_{t \to \infty} \int_0^t e^{-x} dx = \lim_{t \to \infty} -e^{-x} \Big|_0^t = -e^{-\infty} + 1 = 1 \tag{5}$$

If the above limit exists, we say the improper integral converges, if not, then we say it diverges.

We can break up any large improper integral into its component integrals, if any of them are divergent, then the whole integral diverges.

2 First Order Ordinary Differential Equations

2.1 Introduction to ODEs

An ordinary differential equation is one with only ONE independent variable. DEs with multiple independent variables, like $y' = y(x, t)$ are called partial differential equations.

$y' = -\cos t + e^{5+t}$ – FIRST ORDER

$y" + 10y' + 9y = \cos x$ – SECOND ORDER

For any differential equation to classified as linear, any y term must be to the first power. A solution to an ODE is a function, or a set of functions, that satisfies the ODE.

2.2 Separable ODEs

An ODE is classified as separable if it can be expressed as a product of two functions: one in terms of y and one in terms of t.

Here's an example of a separable ODE:

$$\frac{dy}{dt} = -0.02(y - 20) = g(t) \cdot h(y) \tag{6}$$

We can do the illegal 'move dt to the other side' trick and integrate both sides to solve the ODE.

2.3 Linear First Order ODEs

A linear first order ODE can be written in the form $y' + p(t)y = q(t)$. This is called the standard form of the ODE.

Recall that $(\mu y)' = \mu y' + \mu' y$. Thus, we can set $\mu(x)$ as the integrating factor, multiply everything by this IF, and solve the ODE from there.

We can find $\mu(x)$ by taking $\mu(x) = e^{\int p(t) dt}$.

2.4 Modelling with ODEs

Essentially most modelling questions can be put in form of the stuff equation:

$$Change = Ratein - Rateout \tag{7}$$

3 Complex Numbers

My notes for this unit are too messy, so I'll skip them for now and come back to it if I can decipher them.

4 Second Order Ordinary Differential Equations

4.1 Real Roots

Examples order linear ODEs with constant coefficients:
$2y" + 5' - 3y = 0$
$2y" = 5$

Finding Solutions Using the Characteristic Equation

First case: two real roots
Our best case is almost always e^{rt}. So, $y = e^{rt}$ differentiated gives $y = re^{rt}$ differentiated gives $y" = r^2e^{rt}$.

$$2r^2e^{rt} + 5re^{rt} - 3e^{rt} = 0 \tag{8}$$

$$e^{rt}(2r^2 + 5r - 3) = 0 \tag{9}$$

And since we know e^{rt} will never reach zero, we can drop the term entirely. This gives us $r_1 = 0.5$ and $r_2 = -3$.

Hence, we have: $y_1(t) = e^{\frac{1}{2}t}$ and $y_2(t) = e^{-3t}$.

Theorem – If $y_1(t)$ and $y_2(t)$ are solutions to this ODE, and are not constant multiples of each other, all other solutions are $y(t) = c_1y_1(t) + c_2y_2(t)$.

We see from linear algebra that the general solution actually defines a subspace, in this case, a plane.

Second case: one real root
Say we have $y" - 4y' + 4y = 0$: $r^2 - 4r + 4 = 0$ which gives us $r = 2$. Therefore, we have $y(t) = 2e^{rt}$.

Theorem – if $\Delta = 0$ then two solutions to the ODE are given by $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$.

Third case: two complex roots
If we have roots of the form $r = abi$ then we express the general solution as:

$$y = e^{at}\big(c_1\cos(bt) + c_2\sin(bt)\big) \tag{10}$$

4.2 Non-Homogeneous Equations

A non-homogeneous ODE is of the form: $ay" + by' + cy = f(t)$

Essentially we can split the solution $y(t)$ into $u(t) + v(t)$.

$y_c = ay" + by' + cy = 0 \rightarrow$ This is called the complementary solution

$y_p = ay" + by' + cy = f(t) \rightarrow$ This is called the particular solution

We guess the form of the particular solution based on the form of $f(t)$.

4.3 Applications, Over- and Under- Damping

Hooke's Law states that if a spring is stretched or compressed x units of length from its rest position, it will produce a force that opposes the movement that is proportional to x. This force is $F_s = -k\Delta x$.

By Newton's Second Law we can call this: $m\dfrac{d^2x}{dt^2} = F = -kx$. Note that this is a homogeneous second order ODE with constant coefficients, so we can solve it.

Its characteristic equation is $mr^2 + k = 0$ with roots $r = \sqrt{\frac{k}{m}}i$ which gives the general solution: $x(t) = c_1 \cos\sqrt{\frac{k}{m}}t + c_2 \cos\sqrt{\frac{k}{m}}t$. The motion described by this general solution is called **simple harmonic motion**.

$$x(t) = c_1 \cos\sqrt{\frac{k}{m}}t + c_2 \cos\sqrt{\frac{k}{m}}t \tag{11}$$

To make the model more realistic, we need to include some form of damping. Damping force $= -c\frac{dx}{dt}$.

Newton's Second Law gives:

$$m\frac{d^x}{dt^2} = -kx - c\frac{dx}{dt} \iff m\frac{d^x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{12}$$

This is called **damped harmonic motion**.

The motion of a damped oscillator depends on the relative magnitude of the damping force. This means:

- If the damping force is relatively small, the oscillations damp out over time. This is called **underdamping**. (c is smaller compared to m and k)
- If the damping force is so great the mass never oscillates, then it is **overdamped**. (c is large compared to m and k)
- The threshold between these two cases is called **critical damping**.

Case 1: $c^2 - 4mk < 0$ **(underdamping)**

The roots are complex and th general solution is given by:

$$x(t) = e^{-\frac{c}{2m}t}(c_1 \cos\omega t + c_2 \sin\omega t) \tag{13}$$

where

$$\omega = \frac{\sqrt{4mk - c^2}}{2m} \tag{14}$$

In practice, this means that the oscillations will die out but very slowly and over time.

Case 2: $c^2 - 4mk > 0$ **(overdamping)**

In this case r_1 and r_2 are distinct roots, so the general solution is given by:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{15}$$

In practice, overdamping means that the object never oscillates and returns to its equilibrium position immediately.

Case 3: $c^2 - 4mk = 0$ **(critical damping)**

In this case, we have one distinct root, $r_1 = r_2 = \frac{c}{2m}$. The general solution is given by:

$$x(t) = c_1 e^{-\frac{c}{2m}t} + c_2 t e^{-\frac{c}{2m}t} \tag{16}$$

5 Sequences and Series

5.1 Introduction to Taylor Polynomials

Ultimate goal → we will approximate complicated functions using polynomials.

First step:

- Decide where our polynomials should be centred.
- Commit to a point $x = a$ and all of our polynomials will be created using information about the function at this point $x = a$.
- Our approximation should be the "point of perfection."

Second step:

Start constructing the polynomials:

- $p_0(x)$ is a polynomial of the smallest possible degree which agrees with the function $f(x)$ at $x = a$.
  - This implies that $c_0 = f(a)$ and therefore $p_0(x) = f(a)$.
- $p_1(x)$ is a polynomial of the smallest possible degree which agrees with the function $f(x)$ and its derivative $f'(x)$ at $x = a$.
  - This implies $c_0 = f(a)$ and $c_1 = f'(a)$ and therefore $p_1(x) = f(a) + f'(a)(x - a)$.
- $p_2(x)$ is a polynomial of the smallest possible degree which agrees with $f(x)$, $f'(x)$, and $f''(x)$ at $x = a$.
  - This implies that $c_0 = f(a)$, $c_1 = f'(a)$, and $c_2 = f''(a)$ and therefore $p_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.

Taylor Polynomial Formula:

$$p_n(x) = \frac{f^{(n)}(a)}{n!}(x - a)^n \tag{17}$$

5.2 Approximating Errors

We are trying to find the required order of polynomial required to approximate a function (i.e., how many derivatives do we need to consider.) To measure the accuracy of approximating a function value $f(x)$ by the n-th Taylor polynomial $p_n(x)$, you can use the concept of a remainder $R_n(x)$, which is defined to be:

$$R_n(x) = f(x) - P_n(x) \tag{18}$$

We can rewrite this as: $f(x) = R_n(x) + P_n(x)$. The absolute value of $R_n(x)$ is called the error associated with the approximation using the nth Taylor polynomial.

Taylor's Theorem

Overall, we expect our error to depend on three things:

- The general shape of the function (how wild it is)
- The centre a that we choose
- The order of the Taylor polynomial

Taylor's Theorem: Choose a centre $x = a$. If $f(x)$ has $(n+1)$ continuous derivatives on an interval around $x = a$, then for any x in that interval, we can write:

$$f(x) = p_n(x) + R_n(x)$$

where $p_n(x)$ is the n-th Taylor Polynomial of $f(x)$ centred at $x = a$, and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$$

for some c in the interval.

5.3 Sequences and Series

Given a sequence $\{a_n\}$, if the terms become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is convergent and L is the limit of the sequence. The sum of the first k terms is denoted as S_k. This value depends on how many terms we choose to sum over.

Infinite Sum:

An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

For a given k, the sequence of partial sums of the infinite series is the sequence $\{S_k\}$, where:

$$S_k = \sum_{n=1}^{k} a_n$$

5.4 Power Series Convergence

A power series centred at $x = 0$ takes the form:

$$\sum_{n=0}^{\infty} c_0 x^n = c_0 + c_1 x + c_2 n^2 + ... \qquad (19)$$

A series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ... \qquad (20)$$

is a geometric series with ratio $r = x$, we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

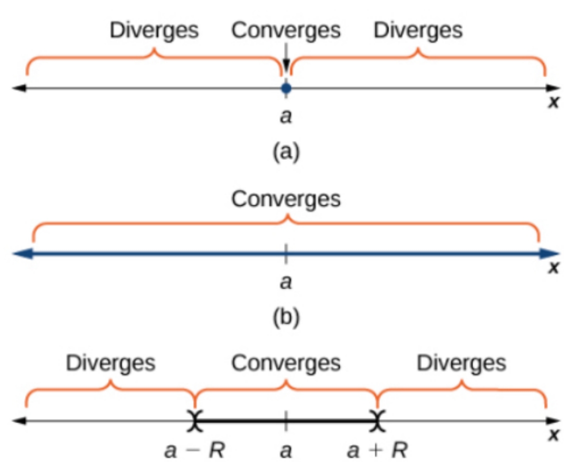

Figure 2: Divergence and Convergence

Ratio Test

To determine the radius of convergence of a power series, we apply the ratio test. This test gives us an interval over which the power series is convergent.

If $\sum_{n=0}^{\infty} a_n$ is a series of nonzero terms (meaning $a_n \neq 0$ for all n), let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then:

1. If $\rho < 1$ then the series converges.
2. If $\rho > 1$ then the series diverges.
3. If $\rho = 1$, the test is inconclusive.

For a series centred at a point $x - ax = ac|x - a| < c$, we can say that the series is convergent for a radius c. Essentially, we have $|x - a| < c$.

Note: the Ratio Test does not say anything about whether the series converges at the endpoints of the interval. We know that inside, the series converges, and outside, the series diverges, but the ratio test is inconclusive for the endpoint of the interval itself.

5.5 Power Series

We already know that **the larger** N is, **the better the approximation given by** $p_N(x)$ **is.**

If we take N to ∞, then we have a power series! Specifically, for $f(x) = e^x$, the power series for when x is centred at $x = 0$ is called the Taylor Series (or Taylor Expansion) of e^x at $x = 0$. This series converges for all Real Numbers.

We want to show that for any value of x, the power series actually converges to $f(x) = e^x$. Meaning, we want to prove the following:

$$\lim_{N\to\infty} p_N(x) = e^x \tag{21}$$

If that is the case, then we can write for that value of x:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \tag{22}$$

We know that $f(x) = e^x =$ (taylor polynomial) + (remainder/error) $= p_N(x) + R_N(x)$. So we need to show that $\lim_{N\to\infty} R_N(x) = e^x$.

Using Taylor's Remainder Theorem

We know, using Taylor's Remainder Theorem, that:

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1} = \frac{e^c}{(N+1)!}3^{N+1} \quad \textbf{for some c between 0 and x.} \tag{23}$$

The limit of the above expression taken to infinity goes to 0.

5.6 Integrating and Differentiating Power Series

Using power series makes working with complicated functions much easier, by turning functions into infinite degree polynomials.

Sometimes finding Taylor series can be cumbersome...

Suppose we wanted to expand $f(x) = \frac{1}{1+x^2}$.

The first two derivatives are:

$$f'(x) - \frac{-2x}{\left(1+x^2\right)^2} \qquad\qquad f''(x) = \frac{6x^2 - 2}{\left(1+x^2\right)^3} \tag{24}$$

... but sometimes we can build Taylor series from already known ones!

Recall:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \textbf{ for } |x| < 1 \tag{25}$$

We can substitute anything we want for x^n:

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \quad \textbf{for} \quad |x^2| < 1 \tag{26}$$

We can expand this to make:

$$\frac{1}{1+x^2} = \sum (-1)^n x^{2n} \tag{27}$$

Theorem: On its interval of convergence, a power series can be differentiated term by term and integrated term by term.

5.7 Integrals with Taylor Series

Let's say for example, that we need find the area under the curve of the normal distribution, which is given by:

$$f(x) = \frac{1}{2\pi} e^{-x^2/2} \tag{28}$$

However, we cannot find an explicit expression for the antiderivative of $f(x)$. Recall that for all for real numbers x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{29}$$

We can replace x with $-x^2$:

$$e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \tag{30}$$

We can now rewrite our integral as:

$$\int_0^1 e^{-x^2/2} dx = \int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \right) dx \tag{31}$$

From here we can integrate term by term.

NOTE: The error when estimating an alternating series is bounded by the first term that you skip.

6 Vector Values Functions

6.1 Parametric Equations

If f and g are continuous functions of the variable t (called a parameter), then:

$$x = f(t) \quad \textbf{and} \quad y = g(t) \tag{32}$$

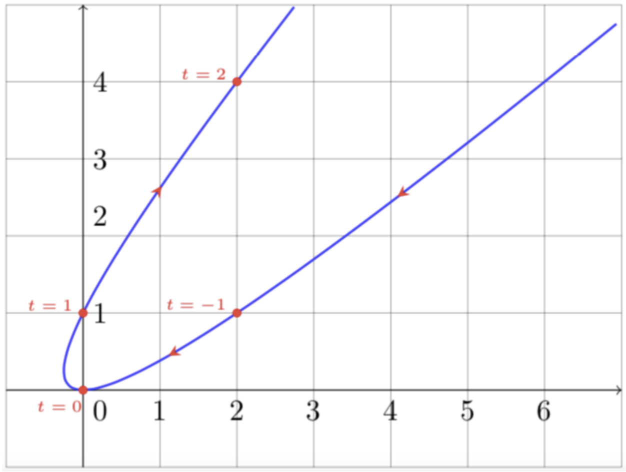

Figure 3: Parametric Equation

The direction of the plane curve as the parameter increases is called the **orientation of the curve** and can be represented by arrows drawn along the curve. We can even put arrows on a graph to show the direction or orientation of the set of parametric equations.

One very important thing to note about parametric equations is that more than one pair of parametric equations can represent the same curve. When the parameter is time, different parametric equations can be used to trace the same curve at different speeds or in different directions.

6.2 Polar Coordinates

Rectangular coordinates are x-y coordinates, or i-j-k coordinates. We can also use **polar coordinates**, which specifies a direction to walk in (θ) and a distance to walk (r). These coordinates are given in (r, θ).

How to go from polar to rectangular coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

How to go from rectangular to polar coordinates:

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \tan(\theta) = x/y \end{cases}$$

Curves in Polar Coordinates

We say that lines are native in rectangular coordinates, while circles are native to polar coordinates.

$$r\sin(\theta) = 3 \iff r = \frac{3}{\sin\theta} \tag{33}$$

6.3 Calculus in Polar Coordinates

There are three main calculus operations we would like to perform with polar coordinates.

1. Find the tangent line at a point.

2. Find the area bounded by a given graph.
3. Find the arclength of a given graph.

To differentiate a parametric equation:

1. For a parametric equation in which we are given $x(t)$ and $y(t)$, we have:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \tag{34}$$

2. For a point described as $(r(\theta), \theta)$, the rectangular coordinates will be:

$$\begin{cases} x = r\theta) \cos(\theta) \\ y = r\theta) \sin(\theta) \end{cases}$$

3. The equation of a line with slope m and a point (a,b) is:

$$y - b = m(x - a) \tag{35}$$

So, once everything has been put together, we get:

$$y(\theta) = r(\theta) \sin\theta \quad \textbf{and} \quad x(\theta) = r(\theta) \cos\theta \tag{36}$$

which gives

$$\frac{dy}{d\theta} = r'(\theta) + r(\theta) \cos\theta \quad \textbf{and} \quad \frac{dx}{d\theta} = r'(\theta) \cos\theta - r(\theta) \sin\theta \tag{37}$$

And so, the slope of the derivative at any point is:

$$\frac{dy}{dx} = \frac{r'(\theta) \sin\theta + r(\theta) \cos\theta}{r'(\theta) \cos\theta - r(\theta) \sin\theta} \tag{38}$$

6.4 Vector-Valued Functions

Our first step is to define a vector function as:

Calculus of vector-valued functions

The limit of a vector-valued function can be determined as:

$$\lim_{t \to a} r(t) = < \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) > \tag{39}$$

The derivative of a vector-valued function can be determined as:

$$r'(t) = < f'(t), g'(t), h'(t) > \tag{40}$$

The integral of a vector-valued function can be determined as:

$$\int r(t)dt = \left\langle \int f(t)dt, \int g(t)dt, \int h(t)dt \right\rangle \tag{41}$$

6.5 Arc Length

There are two ways to specify progress on a route:

1. How long (how much time) one has been travelling, and
2. How far one has travelled.

Given a vector-valued function, we have:

$$L = \int_{t_1}^{t_2} \sqrt{(f'(t))^2 + (g'(t))^2} dt = \int_{t_1}^{t_2} ||r'(t)|| dt \tag{42}$$

6.6 Normal Vector

The unit tangent vector is given by:

$$T(t) = \frac{r'(t)}{||r'(t)||} \tag{43}$$

The principal normal unit vector is a vector that points in the direction a curve is turning. We start by finding the derivative of the unit tangent vector, $T'(t)$:

$$N(t) = \frac{T'(t)}{||T'(t)||}, T'(t) \neq 0 \tag{44}$$

The principal normal unit vector is the only vector that points in the direction that the curve is turning.